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NETWORK OPTIMIZATION BY CONTINUOUS EQUIVALENCE METHOD

by

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A THESIS

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "Network Optimization by Continuous Equivalence Method" submitted by Adegboyega A. Adewunmi in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

In 1964, Schoeffler [15] developed the theory of continuous equivalent network by applying a continuous transformation matrix to the nodal admittance matrix of a given passive network. He applied the concept to minimize the sensitivity of a network function to variations in the element values [16]. More recently other papers on this subject have appeared [2,9]. All these authors have used the steepest descent method to generate the equivalent networks. But the detail of the computational procedure is not described.

In this thesis the concept of continuous equivalence is used to optimize networks with respect to two performance criteria, namely, distribution of elements and sensitivity. The transformation matrix is applied to the loop impedance matrix instead of the nodal admittance matrix as done by Schoeffler in his original work. One of the features of the method of solution adopted by Schoeffler and others is the need to assign new values for the elements of the transformation each time the generating equations are solved. Since these element values are arbitrary this task could be tedious. Although the steepest descent method is also used in this thesis, a way has been found by means of a substitution, which enables the computer to automatically assign the appropriate new values to the elements of the transformation matrix each time the generating equations are solved. Numerical examples of one-port and two-port networks are solved. Certain limitations of Schoeffler's continuous equivalence concept are discussed at the end of this thesis.

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$\underline{A}$	Cauer transformation matrix
$\underline{\mathcal{A}}$	The transpose of $\underline{A}$
$\underline{A}_1$	Modified Cauer transformation matrix
$\underline{B}$	Branch-mesh incidence matrix
e_K	The K^{th} element R, L or C
$\underline{I}$	Vector of mesh currents
S_K	The sensitivity of a network function to changes in the element values
S	Laplace transformation variable
$\underline{T}$	Continuous transformation matrix or network function
$\underline{T}_1$	Modified continuous transformation matrix
t	Transpose
$\underline{V}$	Vector of voltage sources
x	Independent variable
Δx	Incremental value of x
Z(s)	Driving point impedance
$Z_b(s)$	Branch impedance matrix
Φ	Performance criterion

Chapter 1

EQUIVALENT NETWORKS

1-1 <u>Definition:</u> Two networks N_1 and N_2 are said to be equivalent if the driving point and/or transfer immittance of N_1 is the same as that of N_2. The values of some or all the elements of N_1 may be different from those of N_2. This definition will be used in this thesis. Several other connotations can be given to the word "equivalence." We will also be concerned only with passive networks. Fig. 1-1 represents two networks, whose driving point impedances are $Z_a(s)$ and $Z_b(s)$ respectively. s is the Laplace Transform variable.

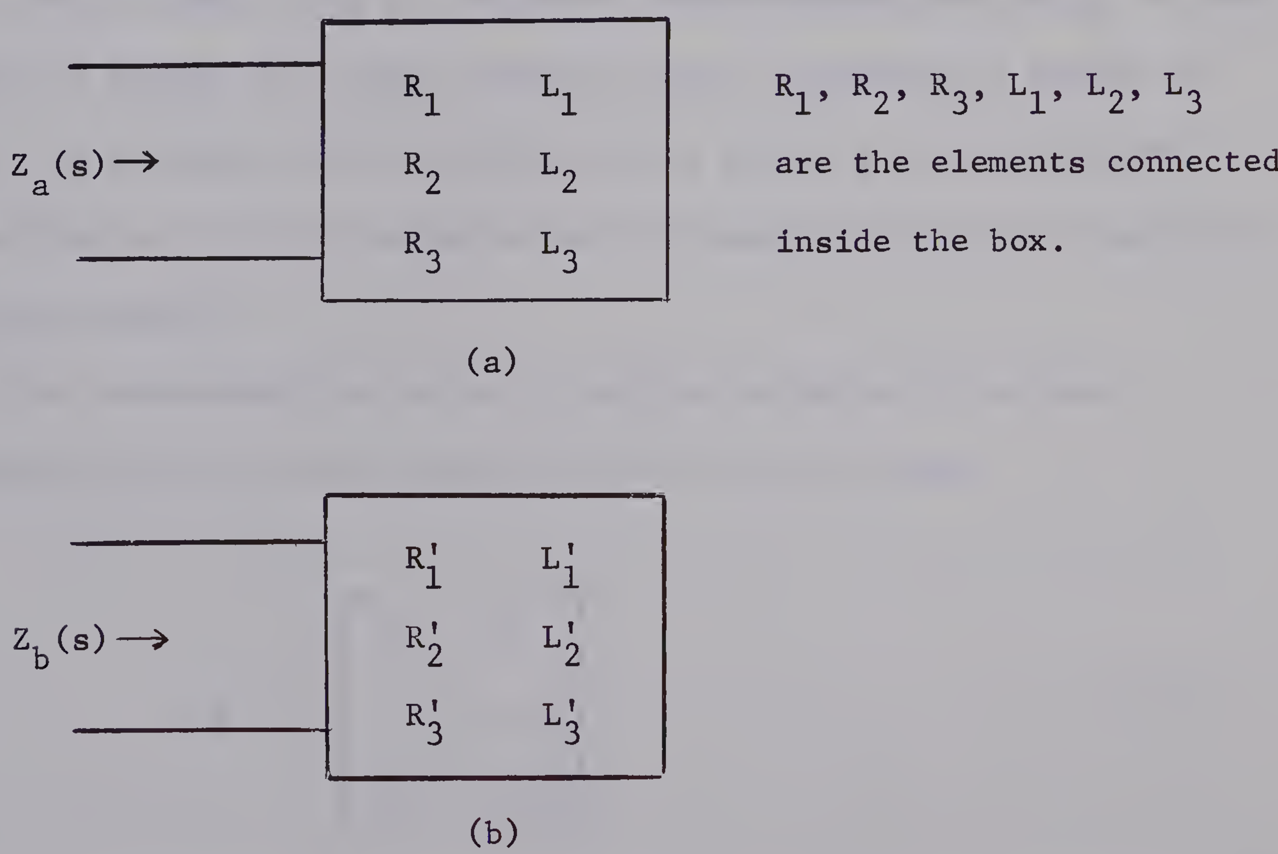


Fig. 1-1 Two one-port passive networks.

These two networks will be equivalent if

$$Z_a(s) = Z_b(s).$$

1-2 Howitt's Approach.

Howitt [5][†] propounded the theory of equivalent networks in 1931. But two years before him Cauer [4] had already propounded the same theory. Their philosophy is based on a long known "fact" but to which nobody before them paid any serious attention. The "fact" is, for a given network with specified values of elements, one and only one impedance function can be realized; but for a given impedance function, there may be an infinite number of networks that can be generated.

Howitt uses a form of congruent transformation matrix $\underline{A}$,[††] on the loop impedance matrix of a given network thereby generating a family of networks. The elements of the transformation matrix $\underline{A}$ are picked arbitrarily so that all the new networks have the same driving point impedance as the given network.

The transformation matrix $\underline{A}$, will be referred to as Cauer transformation matrix in this thesis, and it is of the form

$$\underline{A} = \left[\begin{array}{c|c} \underline{u} & \underline{o} \\ \hline \underline{a}_1 & \underline{a}_2 \end{array}\right] \qquad \ldots\ldots \text{(1-1)}$$

[†] Numbers in square brackets refer to articles or books listed under Bibliography.

[††] Symbols with a bar underneath refer to vectors or matrices as the case may be.

in order to keep the driving point impedance at certain ports constant.

$\underline{u}$ is the identity or unit matrix.

The choice of $\underline{A}$ rests on a theorem which follows:

Theorem.* Two (n + m) x (n + m) loop impedance matrices $\underline{Z}_i(s)$ and $Z_f(s)$ have the same n x n terminal-impedance matrix $\underline{Z}(s)$ if

$$\underline{Z}_f = \underline{\mathcal{A}}\,\underline{Z}_i\,\underline{A} \qquad \text{..... (1-2)}$$

with

$$\underline{A} = \begin{array}{cc} \left[\begin{array}{c|c} \underline{u}_n & \underline{o}_{nxm} \\ \hline \underline{a}_1 & \underline{a}_2 \end{array}\right] & \begin{array}{l} \}n \\ \}m \end{array} \\ \begin{array}{cc} \underbrace{}_{n} & \underbrace{}_{m} \end{array} & \end{array}$$

$$\underline{\mathcal{A}} = \begin{array}{cc} \left[\begin{array}{c|c} \underline{u} & \underline{\alpha}_1 \\ \hline \underline{o}_{mxn} & \underline{\alpha}_2 \end{array}\right] & \begin{array}{l} \}n \\ \}m \end{array} \\ \begin{array}{cc} \underbrace{}_{n} & \underbrace{}_{m} \end{array} & \end{array}$$

and $\underline{\mathcal{A}}$ is non-singular.

Note that $\underline{\mathcal{A}}$ can be made equal to $\underline{A}^t$; the transpose of the matrix $\underline{A}$.

In his work, Howitt makes $\underline{A}$ a constant matrix. And by choosing $\underline{a}_1$, $\underline{a}_2$ in some arbitrary fashion he arrives at a family of networks some

---

*Proof of this theorem is in Appendix I.

of which have the same number of elements as the given network and some of which have fewer elements.

1-3 Schoeffler's Theory of Continuously Equivalent Networks

Recently, Schoeffler [15] has improved on Howitt's theory by making the transformation matrix a function of some dummy variable x and then applying the transformation directly to the elements of the network. This results in the elements of the network being functions of x. At each value of x, the new network is equivalent to the original or given network.

Schoeffler developed his theory for the admittance case (i.e., all elements are connected in parallel). A summary of this theory is given in Appendix II. In practice, networks with elements connected in series occur frequently. Direct application of Schoeffler's results is not found to be very convenient in view of the large number of equations which may have to be solved. Hence it is found desirable to derive parallel results for the impedance case; i.e., the case in which the elements of the network are connected in series. This derivation, which will be presented below, follows Schoeffler's development very closely; the difference occurs in the type of restrictions placed on the transformation matrix, $\underline{T}$.

Suppose we are given a network which can be described by the equation:

$$\underline{V} = \underline{Z}\,\underline{I} \qquad \text{..... (1-3)}$$

where

$$\underline{V} = [V_1, V_2, \text{.....} V_n]^t \qquad \text{..... (1-4)}$$

$$\underline{I} = [I_1, I_2, \text{.....} I_n]^t \qquad \text{..... (1-5)}$$

and $\underline{Z}$ is the n x n loop or mesh impedance matrix.

The branch impedance matrix $\underline{Z}_b$ of the network in general is of the form

$$\underline{Z}_b(s) = \underline{R}_b + s\underline{L}_b + \frac{1}{s}\underline{C}_b^{-1} \qquad \text{..... (1-6)}$$

where $\underline{R}_b$ is the branch resistance matrix

$\underline{L}_b$ is the branch inductance matrix

$\underline{C}_b^{-1}$ is the branch inverse capacitance matrix

If we assume that there is no coupling between any two elements, $\underline{R}_b$, $\underline{L}_b$ and $\underline{C}_b^{-1}$ are all diagonal matrices. For example:

$$\underline{R}_b = \begin{bmatrix} R_1 & 0 & \cdots & \cdots & 0 \\ 0 & R_2 & \cdots & \cdots & 0 \\ \vdots & & R_3 & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & R_m \end{bmatrix}$$

where m is the number of branches in the network.

The loop impedance matrix $\underline{Z}$ can be expressed in terms of the branch impedance matrix by the equation:

$$\underline{Z} = \underline{B}^t \underline{Z}_b \underline{B} \qquad \text{..... (1-7)}$$

where $\underline{B}$ is the branch-mesh incidence matrix and the elements of $\underline{B}$ are defined by

$$b_{jk} = \begin{cases} 1 & \text{If branch } j \text{ is traversed by mesh } k \text{ in the branch direction.} \\ -1 & \text{If branch } j \text{ is traversed by mesh } k \text{ in the opposite direction.} \\ 0 & \text{If branch } j \text{ is not traversed by mesh } k. \end{cases}$$

Using Cauer transformation $\underline{A}$, on the loop impedance matrix $\underline{Z}$, we get another loop-impedance matrix $\underline{Z}'$ given by:

$$\underline{Z}' = \underline{A}^t \, \underline{Z} \, \underline{A} \qquad \text{..... (1-8)}$$

Substituting equation (1-7) into (1-8) we get

$$\underline{Z}' = \underline{A}^t \, \underline{B}^t \, \underline{Z}_b \, \underline{B} \, \underline{A} \qquad \text{..... (1-9)}$$

If a transformation $\underline{T}$ is now applied to the branch impedance matrix $\underline{Z}_b$, we get a new branch impedance matrix $\underline{Z}'_b$ given by

$$\underline{Z}'_b = \underline{T}^t \, \underline{Z}_b \, \underline{T} \qquad \text{..... (1-10)}$$

A new loop impedance matrix $\underline{Z}'$ can also be formed from equation (1-10) using equation (1-7).

Thus:

$$\underline{Z}' = \underline{B}^t \, \underline{T}^t \, \underline{Z}_b \, \underline{T} \, \underline{B} \qquad \text{..... (1-11)}$$

For Z' in equations (1-9) and (1-11) to be equal

$$\underline{A}^t \, \underline{B}^t \, \underline{Z}_b \, \underline{B} \, \underline{A} = \underline{B}^t \, \underline{T}^t \, \underline{Z}_b \, \underline{T} \, \underline{B}$$

$$\therefore \qquad \underline{A}^t \, \underline{B}^t = \underline{B}^t \, \underline{T}^t$$

OR ..... (1-12)

$$\underline{B} \, \underline{A} = \underline{T} \, \underline{B}$$

Equation (1-12) describes the restriction that should be placed on $\underline{T}$ if an equivalent network is to result.

Since the branch impedance matrix $\underline{Z}_b$ can be separated into the different element kinds as represented by equation (1-6), it follows that the transformation matrix $\underline{T}$ can be applied to each element kind separately without affecting the overall result.

If we now assume that the transformation matrix $\underline{T}$ is a function of a dummy variable x the resulting elements from the transformation are then functions of x, i.e., $\underline{R}_b = \underline{R}_b(x)$, $\underline{L}_b = \underline{L}_b(x)$, $\underline{C}_b^{-1} = \underline{C}_b^{-1}(x)$.

$$\text{Let} \qquad \underline{T} = \underline{u} + \Delta\underline{T} \qquad \text{..... (1-13)}$$

where $\underline{u}$ is a $b \times b$ unit matrix, b being the number of branches.

$$\text{Then} \qquad \underline{Z}_b' = \underline{T}^t \, \underline{Z}_b \, \underline{T}$$

$$= \underline{Z}_b + \underline{Z}_b \, \Delta\underline{T} + \Delta\underline{T}^t \, \underline{Z}_b + \Delta\underline{T}^t \, \underline{Z}_b \, \Delta\underline{T} \qquad \text{..... (1-14)}$$

$$\text{If we let} \qquad \Delta\underline{T} = \underline{T}_1 \, \Delta x$$

where Δx is a scalar and

$\underline{T}_1$ is of the form (for a four branch network)

$$\underline{T}_1 = \begin{bmatrix} t_1 & t_2 & t_3 & t_4 \\ t_5 & t_6 & t_7 & t_8 \\ t_9 & t_{10} & t_{11} & t_{12} \\ t_{13} & t_{14} & t_{15} & t_{16} \end{bmatrix}$$

then $$\underline{Z}_b' = \underline{Z}_b + \underline{Z}_b\,\underline{T}_1\,\Delta x + \underline{T}_1^t\,\underline{Z}_b\,\Delta x + \underline{T}_1\,\underline{Z}_b\,\underline{T}_1\,\Delta x^2 \qquad \ldots\ldots (1\text{-}15)$$

Now let $$\underline{T}_1 = \underline{T}_1(x), \quad \underline{Z}_b = \underline{Z}_b(x)$$

and

$$\underline{Z}_b' = \underline{Z}_b(x + \Delta x)$$

$\therefore$ equation (1-15) becomes

$$\frac{\underline{Z}_b(x + \Delta x) - \underline{Z}_b(x)}{\Delta x} = \underline{Z}_b(x)\,\underline{T}_1(x) + \underline{T}_1^t(x)\,\underline{Z}_b(x) + \underline{T}_1^t(x)\,\underline{Z}_b(x)\,\underline{T}_1(x)\,\Delta x \qquad \ldots\ldots (1\text{-}16)$$

Taking the limit in equation (1-16) as $\Delta x \to 0$ we have

$$\frac{d\underline{Z}_b}{dx} = \underline{Z}_b\,\underline{T}_1 + \underline{T}_1^t\,\underline{Z}_b \qquad \ldots\ldots (1\text{-}17)$$

The problem now is to choose the elements of $\underline{T}_1$ so that equivalent networks will result. But even before we choose the elements, we have to place a restriction on $\underline{T}_1$ which is similar to that placed on $\underline{T}$.

Substituting equation (1-13) for $\underline{T}^t$ in equation (1-12) yields

$$\underline{A}^t \, \underline{B}^t = \underline{B}^t \, (\underline{u}^t + \underline{T}_1^t \, \Delta x) \qquad \text{..... (1-18)}$$

Cauer Transformation $\underline{A}$ is also written in the form

$$\underline{A} = \underline{u} + \Delta\underline{A} = \underline{u} + \underline{A}_1 \, \Delta x \qquad \text{..... (1-19)}$$

Substituting (1-19) into (1-18) we have:

$$\underline{A}_1^t \, \underline{B}^t = \underline{B}^t \, \underline{T}_1^t \qquad \text{..... (1-20)}$$

Equation (1-20) gives the restriction on $\underline{T}_1$.

A few remarks on the nature of the matrix $\underline{A}_1$ may be appropriate here. Suppose that it is required to keep the driving point impedance at terminal pairs 1 - 1' and 2 - 2' of the two-port network in Fig. 1-2 constant. A look at the proof of the theorem in Appendix I immediately shows that if the driving point impedances at certain terminal pairs are kept constant, the transfer impedances between those pairs of terminals will also be constant.

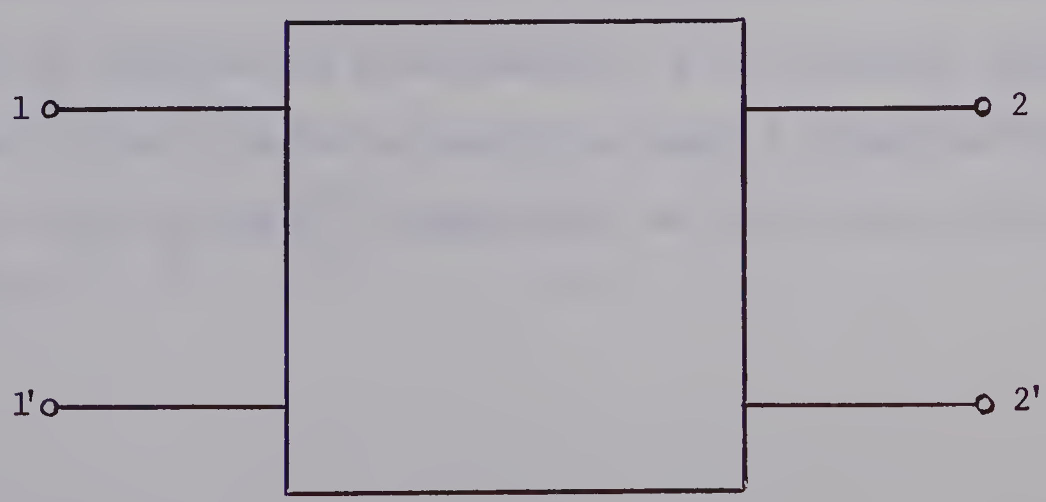


Fig. 1-2 Two-Port Passive Network.

A comparison of equations (1-19) and (A1-3) shows that for the above conditions to be satisfied, the rows of $\underline{A}_1$ corresponding to those terminal pairs must have zero elements.

For example, if there are four loops in Fig. 1-2, then $\underline{A}_1$ will be of the form

$$\underline{A}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{bmatrix} \quad \text{..... (1-21)}$$

The free elements, a_1, a_2, a_3 ..... can be chosen arbitrarily to arrive at equivalent networks. However, it is important to note that these elements should be such that the Cauer transformation matrix, $\underline{A}$, is non-singular. For example in equation (1-21) if $a_3 = -1$, then a_7 must be non-zero, to ensure that $\underline{A}$ is non-singular.

We will conclude this section by arriving at a differential equation for each kind of element in a given network. Equation (1-17) was arrived at by applying the transformation $\underline{T}$ to the branch impedance matrix $\underline{Z}_b$. If we follow a similar procedure and apply a transformation $\underline{T}_{1R}$, $\underline{T}_{1L}$ and $\underline{T}_{1C}$ to $\underline{R}_b$, $\underline{L}_b$ and $\underline{C}_b^{-1}$ respectively, we will arrive at the following equations.

$$\frac{d\underline{R}_b}{dx} = \underline{R}_b \, \underline{T}_{1R} + \underline{T}_{1R}^t \, \underline{R}_b \qquad \text{..... (1-22)}$$

$$\frac{d\underline{L}_b}{dx} = \underline{L}_b \, \underline{T}_{1L} + \underline{T}_{1L}^t \, \underline{L}_b \qquad \text{..... (1-23)}$$

$$\frac{d\underline{C}_b^{-1}}{dx} = \underline{C}_b^{-1} \, \underline{T}_{1C} + \underline{T}_{1C}^t \, \underline{C}_b^{-1} \qquad \text{..... (1-24)}$$

We also obtain a relation similar to equation (1-20):

$$\underline{A}_1^t \, \underline{B}^t = \underline{B}^t \, \underline{T}_{1R}^t = \underline{B}^t \, \underline{T}_{1L}^t = \underline{B}^t \, \underline{T}_{1C}^t \qquad \text{..... (1-25)}$$

The three differential equations (1-22), (1-23) and (1-24), when solved subject to equation (1-25), will generate the element values as functions of x. For different values of x, a family of equivalent networks will result. Therefore, these differential equations will be referred to, as "generating equations" in the rest of this thesis.

If any of the element values should become zero, during the course of the solution, additional constraints should be imposed on the generating equations to prevent the element values from becoming negative. These constraints will also hold if in the given network an element is absent (with respect to the general network) and we want to keep it so. The nature of the additional constraints will be discussed later (Chapter 3, example 3-1).

1-4 Application of Equivalent Networks

An obvious question which comes to one's mind is, Why generate equivalent networks? The need for generating equivalent networks arises from the fact that circuit designers aspire to attain certain goals. Some of these goals include circuits with minimum number of elements, with best possible element distribution and with minimum sensitivity of the immittance function to changes in element values. In other words, circuit designers like to ensure that the circuits they have designed to do a job are the "best" ones.

Hence, instead of choosing the free elements of $\underline{A}_1$ arbitrarily (which amounts to choosing $\underline{T}_1$ arbitrarily) we can choose them to attain one or combinations of the above goals. Thus the main application of the concept of continuous equivalence is in the optimization of networks. However, it can also be used to eliminate coupling between elements as well as to eliminate negative elements from a given network.

1-5 Scope of This Thesis

In this thesis, the theory of continuously equivalent networks will be used to optimize networks. The network optimization will be carried out for two performance criteria, better element distribution and minimum sensitivity. Examples of one- and two-port networks will be considered. The limitations of continuous equivalence as far as this author could see will also be discussed.

One of the contributions of this thesis is in the procedure used to solve the differential equations which result from the application of the

equivalence concept. As will be seen in the next chapter these equations contain arbitrary coefficients. Schoeffler seems to imply that values have to be assigned to these coefficients repeatedly to satisfy certain equations thus leading to solving the differential equations repeatedly. A more systematic method of solving the differential equations is presented in this thesis.

Chapter 2

NETWORK OPTIMIZATION--I (Element Distribution)

2-1 Introduction: As was pointed out in the previous chapter, the continuous equivalence concept enables us to generate a family of networks which have some common characteristic, for instance the same driving point impedance. In obtaining these new networks we may also be interested in improving or optimizing the network with respect to some specific criterion. For example, consider the n-port network, Fig. 2-1.

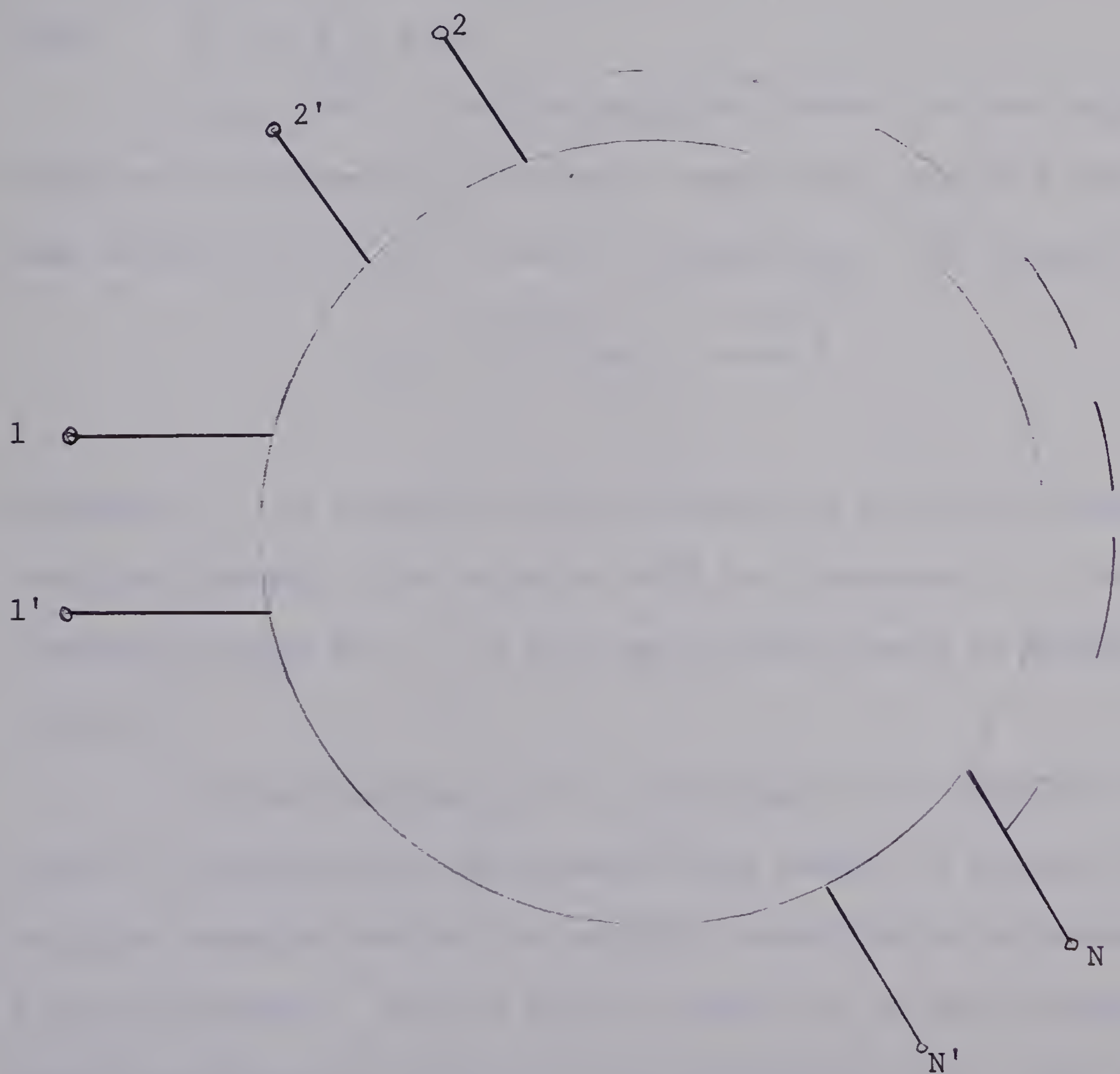


Fig. 2-1 N-Port-Network

Let us assume that it is required to have a better distribution of values in all the elements. As in all optimum problems, this will have to be expressed in a mathematical form. If we represent the criterion by Φ, then it can be expressed as

$$\Phi = \lambda_R \sum_{i=1}^{M} (R_i - \overline{R})^2 + \lambda_L \sum_{i=1}^{N} (L_i - \overline{L})^2 + \lambda_h \sum_{i=1}^{P} (h_i - \overline{h})^2 \qquad \text{..... (2-1)}$$

where $h_i = \frac{1}{C_i}$ and

λ_R, λ_L and λ_h are the weighting factors for the resistances, inductances and inverse capacitances respectively, and $\overline{R}$, $\overline{L}$ and $\overline{h}$ are the mean values of the R's, L's and h's respectively. For instance

$$\overline{R} = \frac{R_1 + R_2 + \ldots\ldots + R_M}{M}$$

Obviously, Φ is a measure of the deviation of the actual element values from their average. Our objective will be to minimize Φ. The actual expression chosen for Φ is only one of several ways of defining the criterion.

In the remaining part of this chapter, two examples will be solved to illustrate the optimization with respect to element distribution. The first example deals with a one-port network while the second deals with a two-port network. For the sake of simplicity, we will attempt to get a better distribution of values for the resistances only. This means that

equation (2-1) simplifies to:

$$\Phi = \sum_{i=1}^{M} (R_i - \overline{R})^2 \qquad \text{..... (2-2)}$$

The examples will be solved in two stages. First, the generating equations for continuous equivalence will be obtained. Then the performance criterion given in equation (2-2) will be minimized using the generating equations as the constraints. The method of steepest descent will be used for arriving at the optimal solution.

2-2 Example 2-1

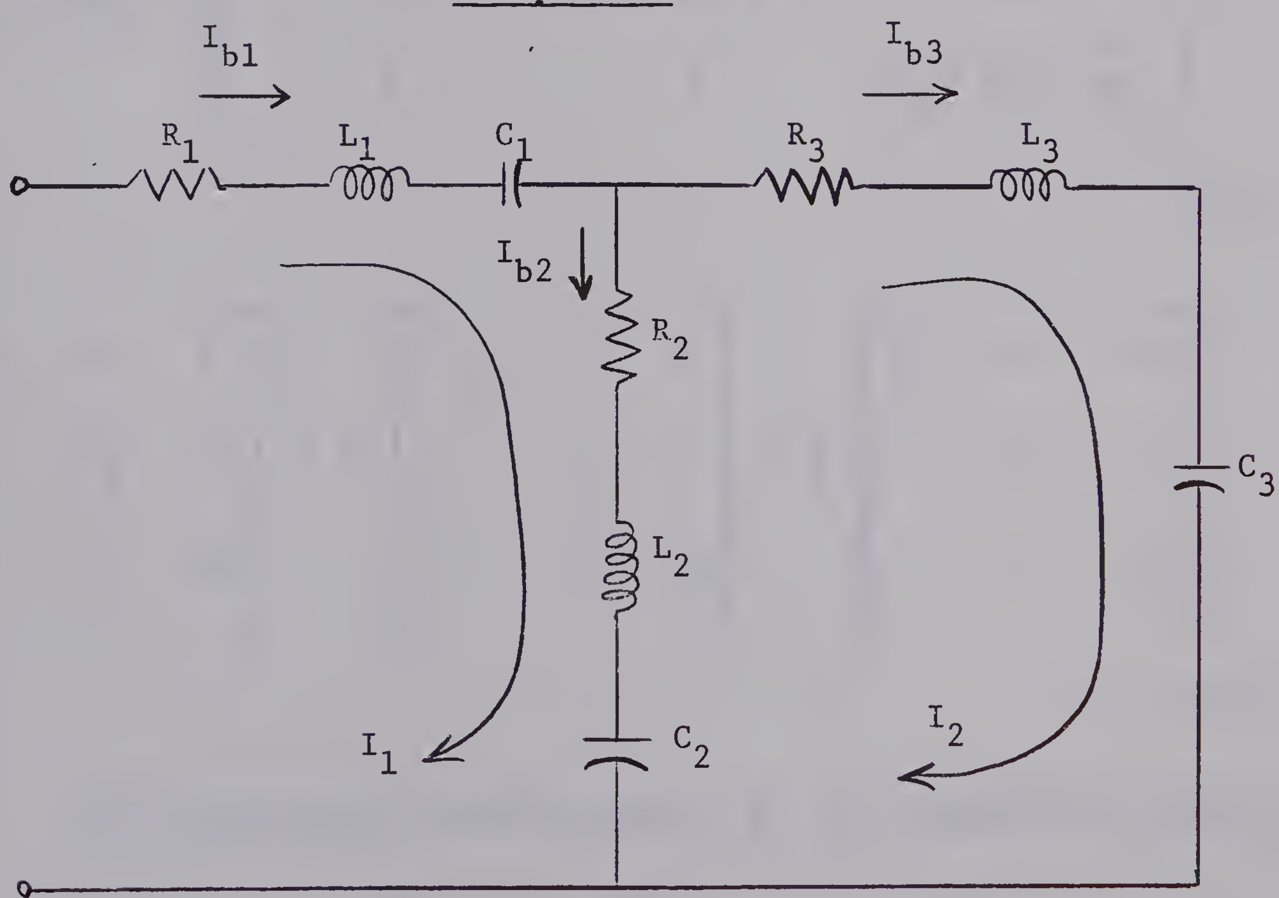
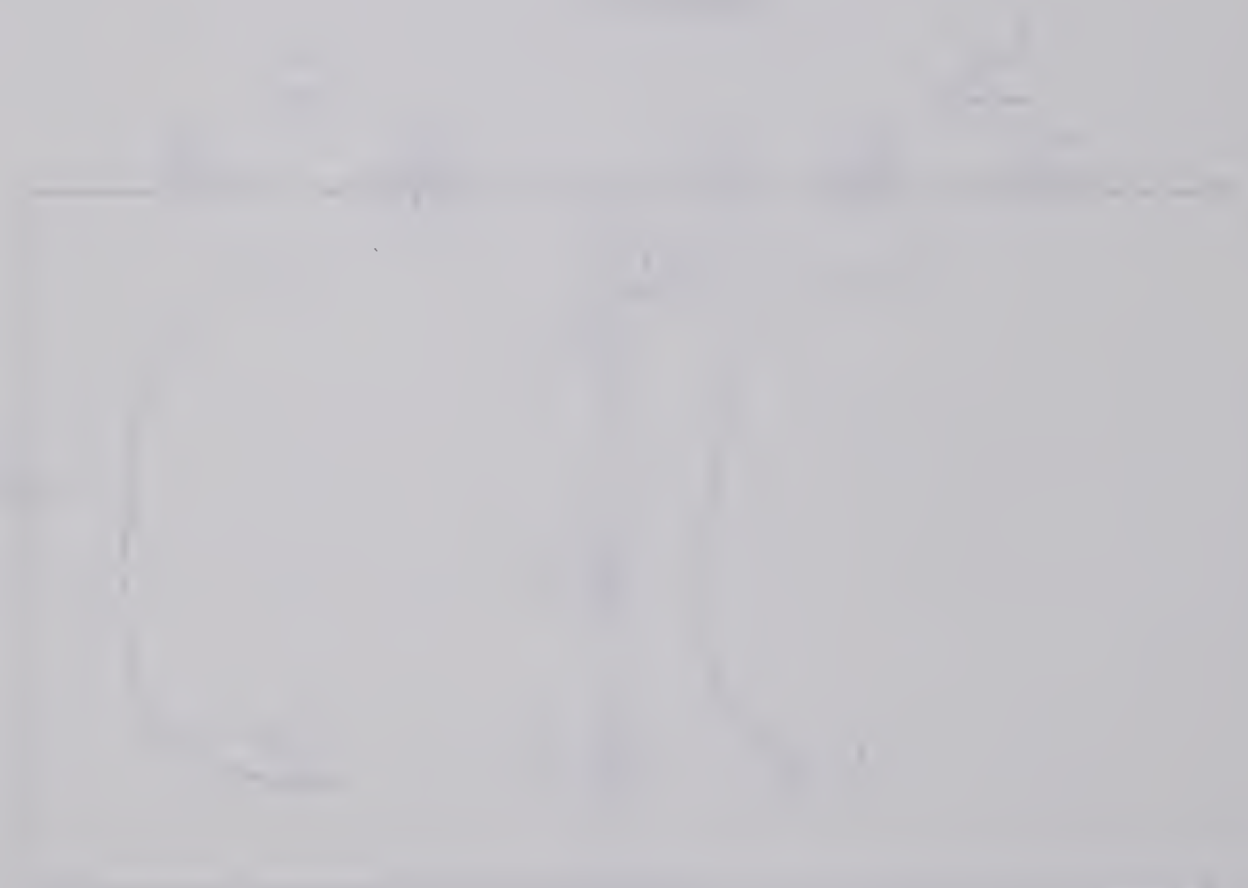


Fig. 2-2 R-L-C-Network.
Dimensions are in ohm, henry and inverse farad or daraf.



Consider the network shown in Fig. 2-2. The values of the elements are:

$$R_1 = 3 \qquad R_2 = 2 \qquad R_3 = 4$$

$$L_1 = 1 \qquad L_2 = 2 \qquad L_3 = 3$$

$$h_1 = 2 \qquad h_2 = 1 \qquad h_3 = 0.5$$

There is no mutual coupling between the inductances.

From Fig. 2-2 we can write the branch impedance as:

$$\underline{Z_b} = \begin{bmatrix} R_1 + sL_1 + \frac{h_1}{s} & 0 & 0 \\ 0 & R_2 + sL_2 + \frac{h_2}{s} & 0 \\ 0 & 0 & R_3 + sL_3 + \frac{h_3}{s} \end{bmatrix} \qquad \text{..... (2-3)}$$

$$= \begin{bmatrix} R_1 & 0 & 0 \\ 0 & R_2 & 0 \\ 0 & 0 & R_3 \end{bmatrix} + s \begin{bmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_3 \end{bmatrix} + \frac{1}{s} \begin{bmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{bmatrix} \qquad \text{..... (2-4)}$$

The branch-mesh incidence matrix $\underline{B}$, by inspection of Fig (2-2) is,

$$B = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \qquad \text{..... (2-5)}$$

The generating equations are the same as equations (1-22), (1-23) and (1-24). For convenience, the equations are written below:

$$\frac{d\underline{R}_b}{dx} = \underline{R}_b \, \underline{T}_1 + \underline{T}_1^t \, \underline{R}_b \qquad \ldots [1\text{-}22]$$

$$\frac{d\underline{L}_b}{dx} = \underline{L}_b \, \underline{T}_{1L} + \underline{T}_{1L}^t \, \underline{L}_b \qquad \ldots [1\text{-}23]$$

$$\frac{d\underline{H}_b}{dx} = \underline{H}_b \, \underline{T}_{1C} + \underline{T}_{1C}^t \, \underline{H}_b \qquad \ldots [1\text{-}24]$$

where $\underline{T}_{1R}$, $\underline{T}_{1L}$ and $\underline{T}_{1C}$ are of the form

$$T_{1R} = \begin{bmatrix} t_1 & t_2 & t_3 \\ t_4 & t_5 & t_6 \\ t_7 & t_8 & t_9 \end{bmatrix} \qquad T_{1L} = \begin{bmatrix} t_1' & t_2' & t_3' \\ t_4' & t_5' & t_6' \\ t_7' & t_8' & t_9' \end{bmatrix}$$

$$\underline{T}_{1C} = \begin{bmatrix} t_1'' & t_2'' & t_3'' \\ t_4'' & t_5'' & t_6'' \\ t_7'' & t_8'' & t_9'' \end{bmatrix}$$

It must be remembered that for equivalence to result equation (1-25) must also be satisfied. In order that the driving point impedance remains invariant under the transformations, $\underline{A}_1$ should be of the form

$$\underline{A}_1 = \begin{bmatrix} 0 & 0 \\ a_1 & a_2 \end{bmatrix}$$

Substituting for $\underline{R}_b$ and $\underline{T}_{1R}$ in equation (1-22) we have:

$$\frac{d}{dx}\begin{bmatrix} R_1 & 0 & 0 \\ 0 & R_2 & 0 \\ 0 & 0 & R_3 \end{bmatrix} = \begin{bmatrix} 2R_1t_1 & R_1t_2+R_2t_4 & R_1t_3+R_3t_7 \\ R_1t_2+R_2t_4 & 2R_2t_5 & R_2t_6+R_3t_8 \\ R_3t_7+R_1t_3 & R_2t_6+R_3t_8 & 2R_3t_9 \end{bmatrix}$$

which gives

$$\frac{dR_1}{dx} = 2R_1t_1$$

$$\frac{dR_2}{dx} = 2R_2t_5 \qquad \text{..... (2-6)}$$

$$\frac{dR_3}{dx} = 2R_3t_9$$

and

$$R_1t_2 + R_2t_4 = 0$$

$$R_1t_3 + R_3t_7 = 0 \qquad \text{..... (2-7)}$$

$$R_2t_6 + R_3t_8 = 0$$

Substituting for $\underline{A}_1$, $\underline{B}$ and $\underline{T}_1$ in equation (1-25) and substituting the

result in equation (2-7) the following relationships are obtained.

$$2R_1t_1 = -R_2(a_2 + a_1) + R_3a_1$$

$$2R_2t_5 = R_2(a_2 - a_1) - R_3a_1 \qquad \text{..... (2-8)}$$

$$2R_3t_9 = R_2(a_2 + a_1) + R_3(2a_2 + a_1)$$

If equation (2-8) is now substituted in equation (2-6) the result is

$$\frac{dR_1}{dx} = -R_2(a_2 + a_1) + R_3a_1$$

$$\frac{dR_2}{dx} = R_2(a_2 - a_1) - R_3a_1 \qquad \text{..... (2-9)}$$

$$\frac{dR_3}{dx} = R_2(a_2 + a_1) + R_3(2a_2 + a_1)$$

Going through a similar procedure for the inductance and capacitance group we have

$$\frac{dL_1}{dx} = -L_2(a_2 + a_1) + L_3a_1$$

$$\frac{dL_2}{dx} = L_2(a_2 - a_1) - L_3a_1 \qquad \text{..... (2-10)}$$

$$\frac{dL_3}{dx} = L_2(a_2 + a_1) + L_3(2a_2 + a_1)$$

$$\frac{dh_1}{dx} = -h_2(a_2 + a_1) + h_3a_1$$

$$\frac{dh_2}{dx} = h_2(a_2 - a_1) - h_3a_1 \qquad \text{..... (2-11)}$$

$$\frac{dh_3}{dx} = h_2(a_2 + a_1) + h_3(2a_2 + a_1)$$

In equations (2-9), (2-10) and (2-11) a_1 and a_2 are not known and can be assigned values arbitrarily. However, in this example, the values will be chosen, keeping in mind that the distribution of values of the resistances must be optimum.

For this example, equation (2-2) becomes

$$\Phi = \sum_{i=1}^{3} (R_i - \overline{R})^2 \qquad \text{... [2-2]}$$

where $\overline{R} = \dfrac{R_1 + R_2 + R_3}{3}$

or $\Phi = \frac{2}{3}(R_1^2 + R_2^2 + R_3^2 - R_1R_2 - R_1R_3 - R_2R_3) \qquad \text{..... (2-12)}$

The method of steepest descent will be used to determine the values of x which ensure that Φ goes rapidly to its minimum or at least approaches its minimum fast.

For Φ to approach its minimum rapidly, it is required that

$\dfrac{d\Phi}{dx} < 0.$

But $$\frac{d\Phi}{dx} = \frac{\partial\Phi}{\partial R_1}\ \frac{dR_1}{dx} + \frac{\partial\Phi}{\partial R_2} \cdot \frac{dR_2}{dx} + \frac{\partial\Phi}{\partial R_3} \cdot \frac{dR_3}{dx} \qquad \text{..... (2-13)}$$

Evaluating $\frac{\partial\Phi}{\partial R_1}$, $\frac{\partial\Phi}{\partial R_2}$ and $\frac{\partial\Phi}{\partial R_3}$ by differentiation of equation (2-12) and substituting the results into equation (2-13) and then substituting for $\frac{dR_1}{dx}$, $\frac{dR_2}{dx}$ and $\frac{dR_3}{dx}$ from equations (2-9), (2-10) and (2-11) into equation (2-13), we get

$$\frac{d\Phi}{dx} = \frac{2}{3}\{a_2(-4R_1R_2 - 2R_1R_3 + 2R_2^2 + 4R_3^2)$$
$$+ a_1(-2R_1R_2 + 2R_1R_3 - 2R_2^2 + 2R_3^2)\} \qquad \text{..... (2-15)}$$

Let us define a_1 and a_2 by

$$a_2 = -\frac{2}{3}(-4R_1R_2 - 2R_1R_3 + 2R_2^2 + 4R_3^2) \qquad \text{..... (2-16)}$$

$$a_1 = -\frac{2}{3}(-2R_1R_2 + 2R_1R_3 - 2R_2^2 + 2R_3^2) \qquad \text{..... (2-17)}$$

Thus $$\frac{d\Phi}{dx} = -a_2^2 - a_1^2$$

hence $\frac{d\Phi}{dx}$ is always negative.

By making the definitions described by equations (2-16) and (2-17) we avoid the uncertainty involved in assigning values to the coefficients a_1 and a_2 and repeatedly solving the differential equations until we arrive at the optimum values of a_1 and a_2.

For this <u>choice</u> of values of a_1 and a_2 given by equations (2-16) and (2-17) the linear generating equations (2-9), (2-10) and (2-11) become nonlinear and given by

$$\frac{d\underline{R}_b}{dx} = \underline{M}\,\underline{R}_b \qquad \text{..... (2-18)}$$

$$\frac{d\underline{L}_b}{dx} = \underline{M}\,\underline{L}_b \qquad \text{..... (2-19)}$$

$$\frac{d\underline{H}_b}{dx} = \underline{M}\,\underline{H}_b \qquad \text{..... (2-20)}$$

where

$$\underline{M} = \begin{bmatrix} 0 & \frac{2}{3}(6R_3^2 - 6R_1R_2) & \frac{2}{3}(2R_1R_2 - 2R_1R_3 + 2R_2^2 - 2R_3^2) \\ 0 & \frac{2}{3}(2R_1R_2 + 4R_1R_3 - 4R_2^2 - 2R_3^2) & \frac{2}{3}(2R_1R_3 + 2R_3^2 - 2R_1R_2 - 2R_2^2) \\ 0 & \frac{2}{3}(6R_1R_2 - 6R_3^2) & \frac{2}{3}(10R_1R_2 + 2R_1R_3 - 2R_2^2 - 10R_3^2) \end{bmatrix}$$

and as before

$$\underline{R}_b = [R_1 \quad R_2 \quad R_3]^t$$

$$\underline{L}_b = [L_1 \quad L_2 \quad L_3]^t$$

$$\underline{H}_b = [h_1 \quad h_2 \quad h_3]^t$$

The nonlinear differential equations (2-18), (2-19) and (2-20) are solved on the digital computer (IBM 360). The solutions are displayed in Appendix IV.

Assuming that at $x = 0.030$ we are satisfied with the element values, Table 2-1 shows the comparison between the new element values and the given ones as well as the respective values of Φ.

	ORIGINAL NETWORK $x = 0$	NEW NETWORK $x = 0.030$
R_1	3	2.921
R_2	2	2.790
R_3	4	2.860
L_1	1	1.056
L_2	2	2.664
L_3	3	2.005
h_1	2	2.151
h_2	1	1.186
h_3	0.5	0.215
Φ	2	0.008

TABLE 2-1 Summary of Results

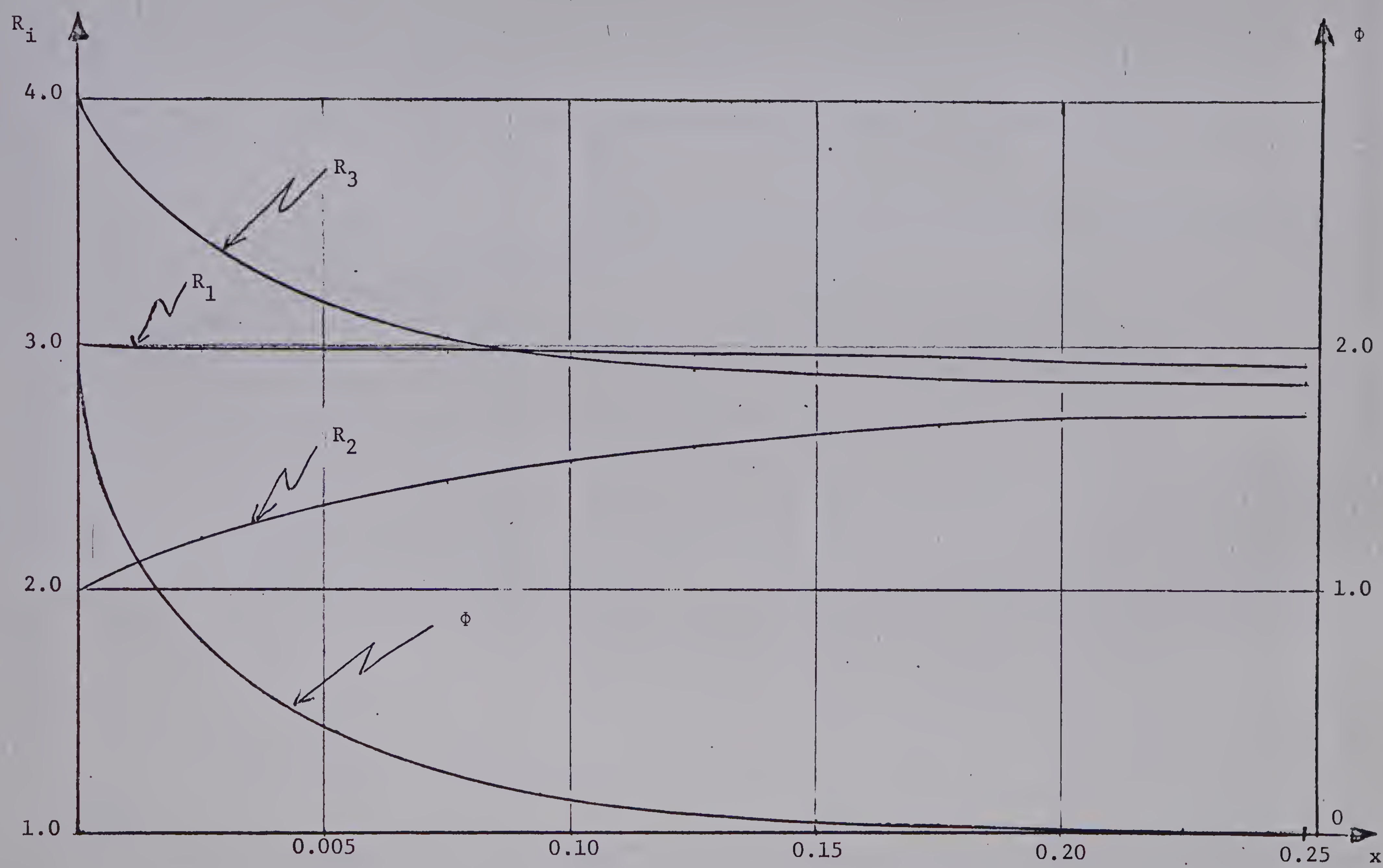


Fig. 2-3 Variation of Resistive Elements with x.

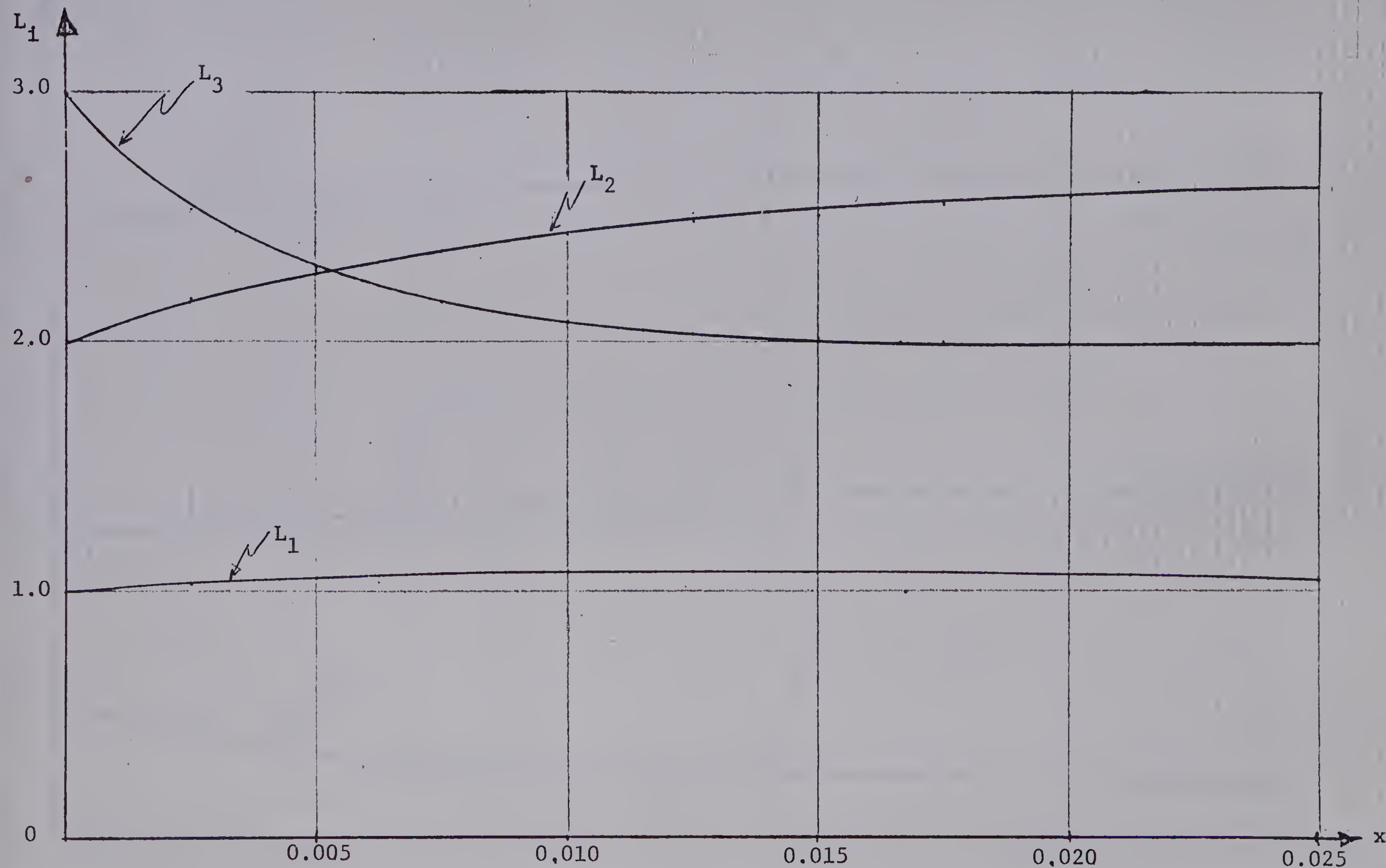


Fig. 2-4 Variation of Inductive Elements with x.

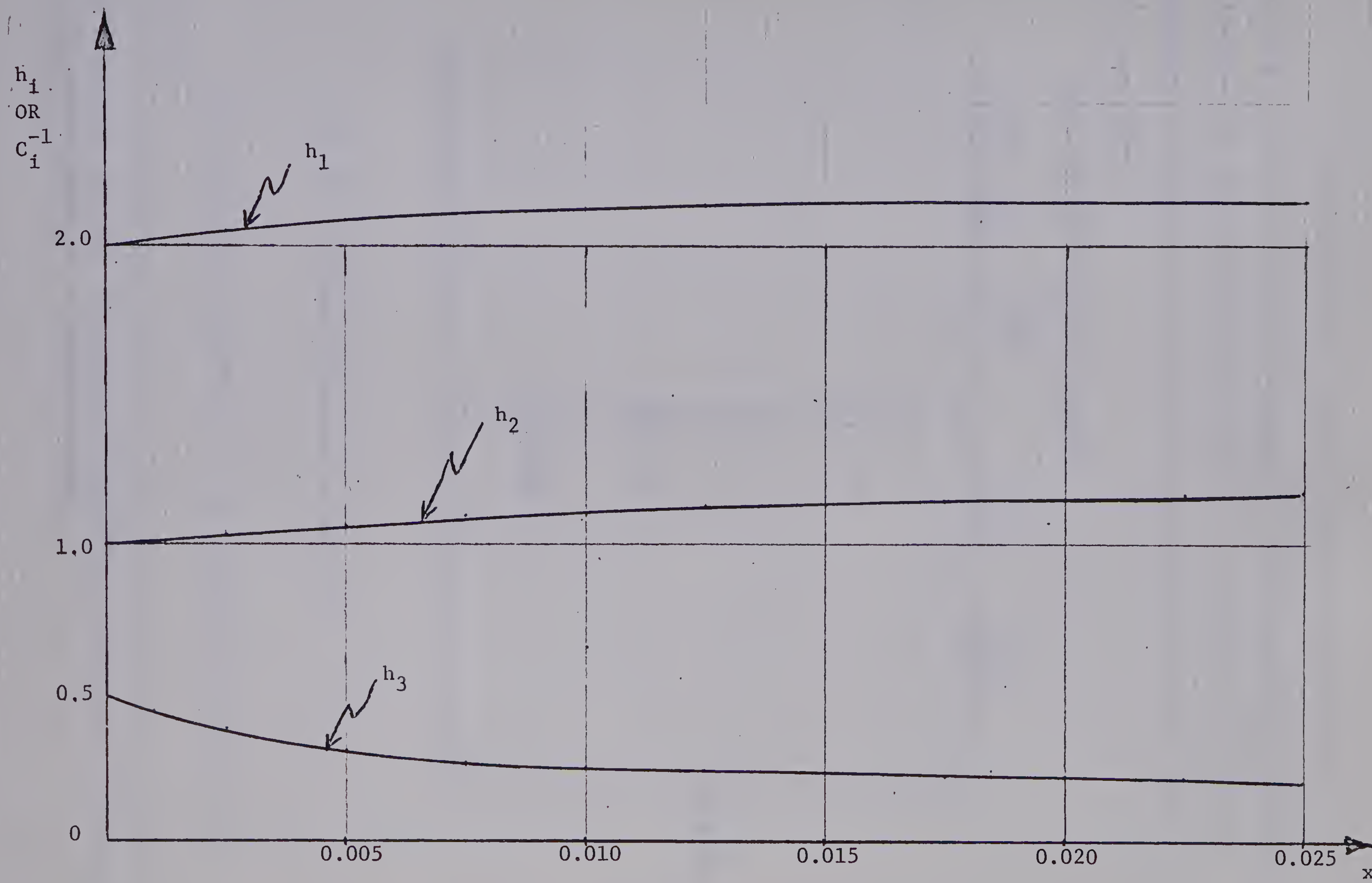


Fig. 2-5 Variation of Capacitive Elements with x.

If we plot the values of each element as a function of x, the resulting curves may be used if necessary to obtain approximate analytical expressions for the elements as functions of x by using curve fitting procedures. In Figs. 2-3, 2-4 and 2-5, the variations of R's, L's and h's with x are shown.

Fig. 2-6 shows the new network.

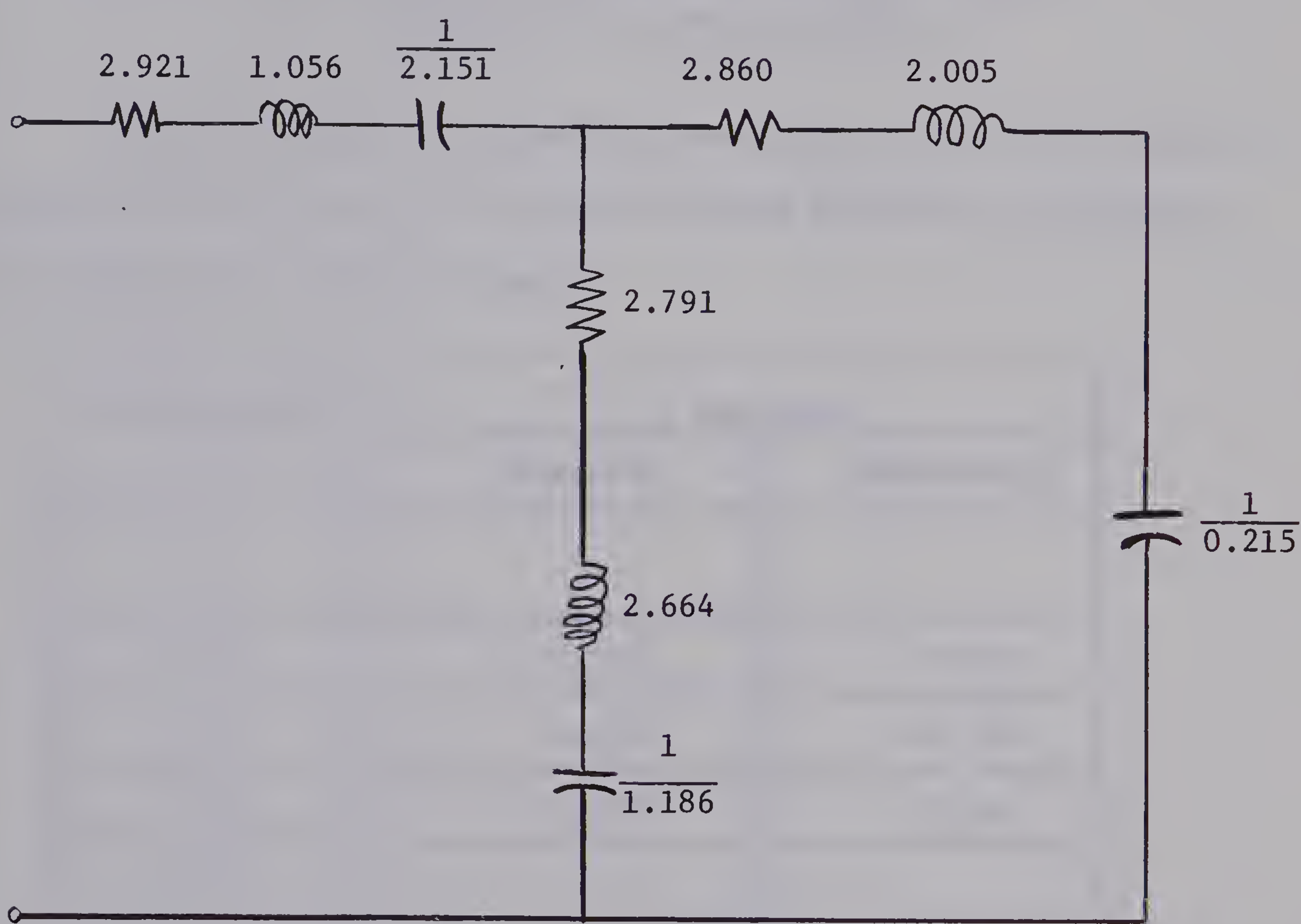


Fig. 2-6 Network Equivalent to Fig. 2-1

The driving point impedance* of the original and the new networks are:

---

*The expression for each Z(s) has been normalized to make the coefficient of the highest power of s equal to unity.

Original network (Fig. 2-1)

$$Z(s) = \frac{s^4 + 3.182\ s^3 + 3.770\ s^2 + 1.953\ s + 0.318}{s(0.455\ s^2 + 0.540\ s + 0.136)}$$

New network (Fig. 2-6)

$$Z(s) = \frac{s^4 + 3.199\ s^3 + 3.690\ s^2 + 1.970\ s + 0.318}{s(0.454\ s^2 + 0.550\ s + 0.137)}$$

Table 2-2 shows the percentage deviation of the coefficients of the powers of s for the Z(s) of the new network from the corresponding coefficients of the original network.

COEFFICIENTS OF	% DEVIATION	
	NUMERATOR	DENOMINATOR
s^4	0	
s^3	- 0.220	0.250
s^2	2.120	- 0.732
s^1	- 0.895	- 0.514
s^0	0	

TABLE 2-2 Percentage Deviation in Coefficients

Example 2-2

2-3 We will now consider the optimization of a 2-port network using the same performance criterion, namely element distribution. The given network is shown in Fig. 2-7. The driving point impedances at both ports will be made invariant under the transformation. The number of loops being greater than the number of ports, is by no means an accident. That this indeed is a requirement follows from the proof of the theorem stated in Chapter 1. This point will be discussed in some detail in Chapter 4.

The branch impedance matrix $\underline{Z}_b(s)$ and the branch-mesh incidence matrix $\underline{B}$, are:

$$\underline{Z}_b(s) = \begin{bmatrix} R_1 & 0 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 & 0 \\ 0 & 0 & R_3 & 0 & 0 \\ 0 & 0 & 0 & R_4 & 0 \\ 0 & 0 & 0 & 0 & R_5 \end{bmatrix} + s \begin{bmatrix} L_1 & 0 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 & 0 \\ 0 & 0 & L_3 & 0 & 0 \\ 0 & 0 & 0 & L_4 & 0 \\ 0 & 0 & 0 & 0 & L_5 \end{bmatrix} \qquad \text{..... (2-21)}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \text{..... (2-22)}$$

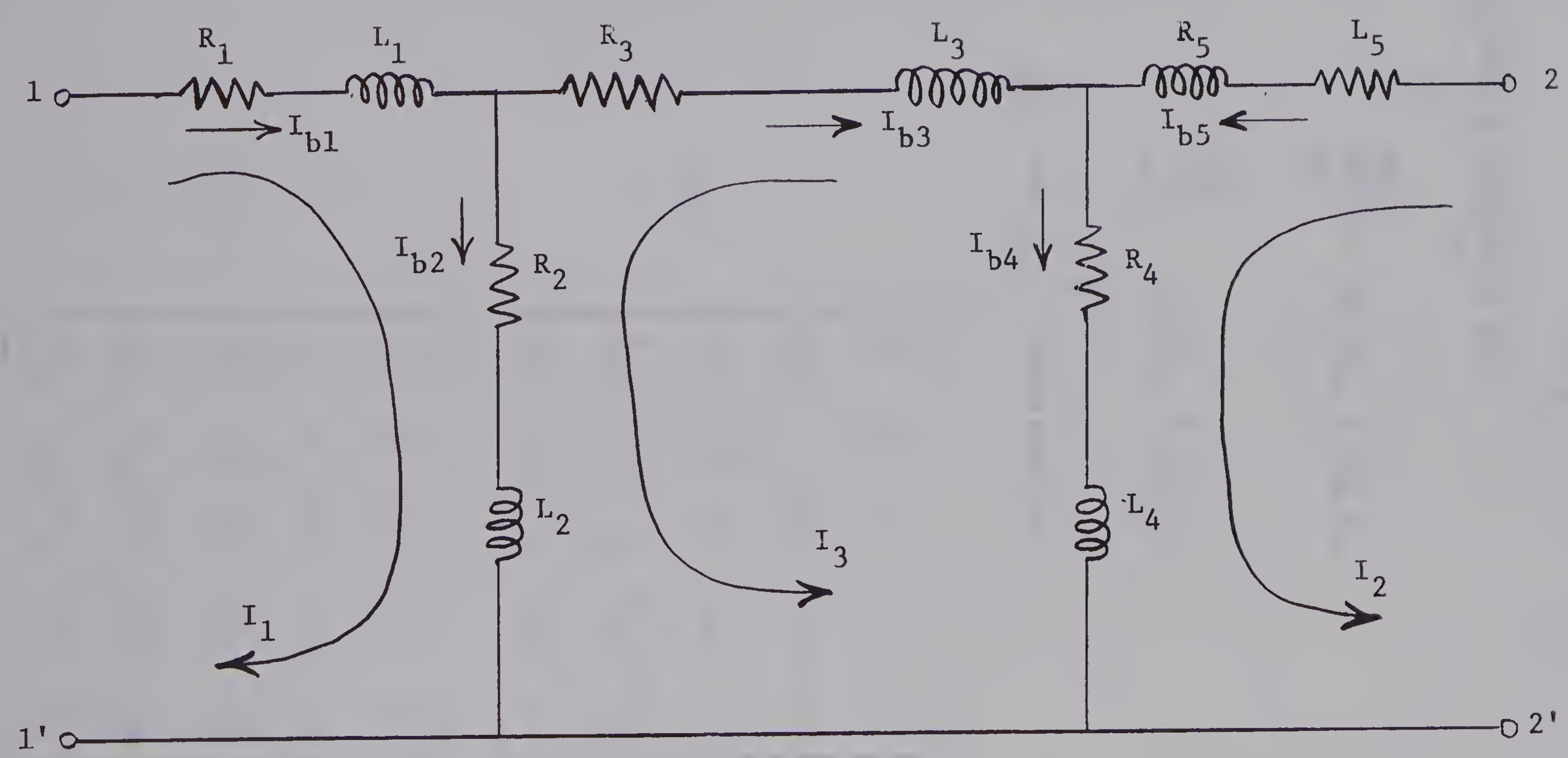


Fig. 2-7 Given R-L Network

ELEMENT VALUES

$R_1 = 3$	$R_2 = 4$	$R_3 = 6$	$R_4 = 3.5$	$R_5 = 7$
$L_1 = 1$	$L_2 = 3$	$L_3 = 5$	$L_4 = 8$	$L_5 = 6$

The matrix $\underline{A}_1$ is given by

$$\underline{A}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_1 & a_2 & a_3 \end{bmatrix} \qquad \ldots\ldots (2\text{-}24)$$

The generating equations are:

$$\frac{d\underline{R}_b}{dx} = \underline{R}_b \, \underline{T}_{1R} + \underline{T}^t_{1R} \, \underline{R}_b \qquad \ldots\ldots (2\text{-}25)$$

$$\frac{d\underline{L}_b}{dx} = \underline{L}_b \, \underline{T}_{1L} + \underline{T}^t_{1L} \, \underline{L}_b \qquad \ldots\ldots (2\text{-}26)$$

where $\underline{T}_{1R}$ and $\underline{T}_{1L}$ are given by

$$\underline{T}_{1R} = \begin{bmatrix} t_1 & t_2 & t_3 & t_4 & t_5 \\ t_6 & t_7 & t_8 & t_9 & t_{10} \\ t_{11} & t_{12} & t_{13} & t_{14} & t_{15} \\ t_{16} & t_{17} & t_{18} & t_{19} & t_{20} \\ t_{21} & t_{22} & t_{23} & t_{24} & t_{25} \end{bmatrix}$$

$$\ldots\ldots (2\text{-}27)$$

$$\underline{T}_{1L} = \begin{bmatrix} t'_1 & t'_2 & t'_3 & t'_4 & t'_5 \\ t'_6 & t'_7 & t'_8 & t'_9 & t'_{10} \\ t'_{11} & t'_{12} & t'_{13} & t'_{14} & t'_{15} \\ t'_{16} & t'_{17} & t'_{18} & t'_{19} & t'_{20} \\ t'_{21} & t'_{22} & t'_{23} & t'_{24} & t'_{25} \end{bmatrix}$$

As before, equation (1-25) must also be satisfied, if equivalent network is to result.

Thus: $$\underline{B}^t\ \underline{T}^t_{1R} = \underline{A}^t_1\ \underline{B}^t = \underline{B}^t\ \underline{T}^t_{1L} \qquad \text{..... (2-28)}$$

Since the procedure used here is the same as that used in example 2-1, most details will be omitted. The important equations and results will be simply stated.

Thus, the generating equations become

$$\frac{dR_1}{dx} = R_2(a_1 - a_2 - a_3) - R_3a_1$$

$$\frac{dR_2}{dx} = R_2(a_1 + a_2 + a_3) + R_3a_1$$

$$\frac{dR_3}{dx} = R_2(a_3 - a_1) + R_3(2a_3 + a_2 - a_1) + R_4(a_3 + a_2)$$

$$\frac{dR_4}{dx} = -R_3a_2 - R_4(a_1 + a_2 - a_3)$$

$$\frac{dR_5}{dx} = R_3a_2 + R_4(a_1 - a_2 - a_3)$$

..... (2-29)

$$\frac{dL_1}{dx} = L_2(a_1 - a_2 - a_3) - L_3a_1$$

$$\frac{dL_2}{dx} = L_2(a_1 + a_2 + a_3) + L_3a_1$$

$$\frac{dL_3}{dx} = L_2(a_3 - a_1) + L_3(2a_3 + a_2 - a_1) + L_4(a_3 + a_2)$$

$$\frac{dL_4}{dx} = -L_3a_2 - L_4(a_1 + a_2 - a_3)$$

$$\frac{dL_5}{dx} = L_3a_2 + L_4(a_1 - a_2 - a_3)$$

..... (2-30)

Again the performance criterion Φ is defined as:

$$\Phi = \sum_{i=1}^{5} (R_i - \bar{R})^2 \qquad ..... (2\text{-}31)$$

where $\bar{R} = \left(\dfrac{R_1 + R_2 + R_3 + R_4 + R_5}{5} \right)$

As in the last example, the method of steepest descent will be utilized to ensure that Φ goes rapidly to its minimum. This requires that $\frac{d\Phi}{dx} < 0$.

But $$\frac{d\Phi}{dx} = \frac{2}{5}(a_1K_1 + a_2K_2 + a_3K_3) \qquad ..... (2\text{-}32)$$

where $$K_1 = (4R_1R_2 - 4R_1R_3 + 4R_2^2 - R_2R_4 - R_2R_5 - 4R_3^2 + R_3R_4 + R_3R_5 - 5R_4^2 + 5R_4R_5)$$

$$K_2 = (-5R_1R_2 - R_1R_3 + R_1R_4 + 5R_2^2 - R_2R_3 + R_2R_4 + 4R_3^2 + 4R_3R_5 - 4R_4^2 + 4R_4R_5)$$

$$K_3 = (-6R_1R_2 - 2R_1R_3 - R_1R_4 + 4R_2^2 + 2R_2R_3 - 2R_2R_4 - R_2R_5$$

$$+ 8R_3^2 + 2R_3R_4 - 2R_3R_5 + 4R_4^2 - 6R_4R_5)$$

If we let

$$a_1 = -K_1$$

$$a_2 = -K_2 \qquad \qquad ..... (2\text{-}33)$$

$$a_3 = -K_3$$

then equation (2-32) becomes

$$\frac{d\Phi}{dx} = \frac{2}{5}(-a_1^2 - a_2^2 - a_3^2)$$

It is easy to see that this is always negative.

Substituting for a_1, a_2 and a_3 in equations (2-29) and (2-30) using equation (2-33) the result in vector form is:

$$\frac{d\underline{R}_b}{dx} = \underline{K}\,\underline{R}_b \qquad \qquad ..... (2\text{-}34)$$

$$\frac{d\underline{L}_b}{dx} = \underline{K}\,\underline{L}_b \qquad \qquad ..... (2\text{-}35)$$

where

$$\underline{K} = \begin{bmatrix} 0 & K_3 + K_2 - K_1 & K_1 & 0 & 0 \\ 0 & -(K_1 + K_2 + K_3) & -K_1 & 0 & 0 \\ 0 & K_1 - K_3 & K_1 - K_2 - 2K_3 & -(K_2 + K_3) & 0 \\ 0 & 0 & K_2 & K_1 + K_2 - K_3 & 0 \\ 0 & 0 & -K_2 & K_2 + K_3 - K_1 & 0 \end{bmatrix}$$

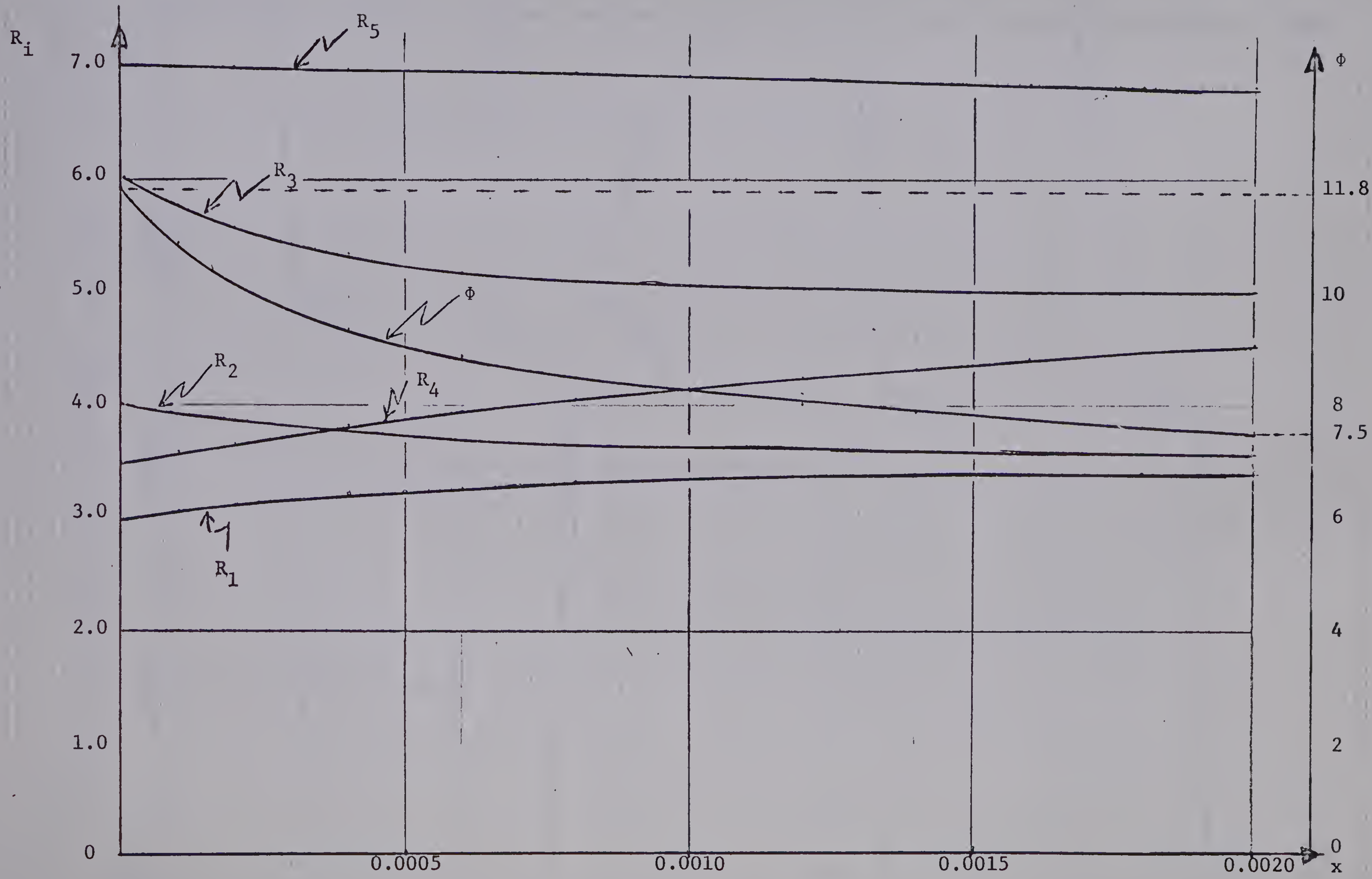


Fig. 2-8 Variation of Resistive Elements with x.

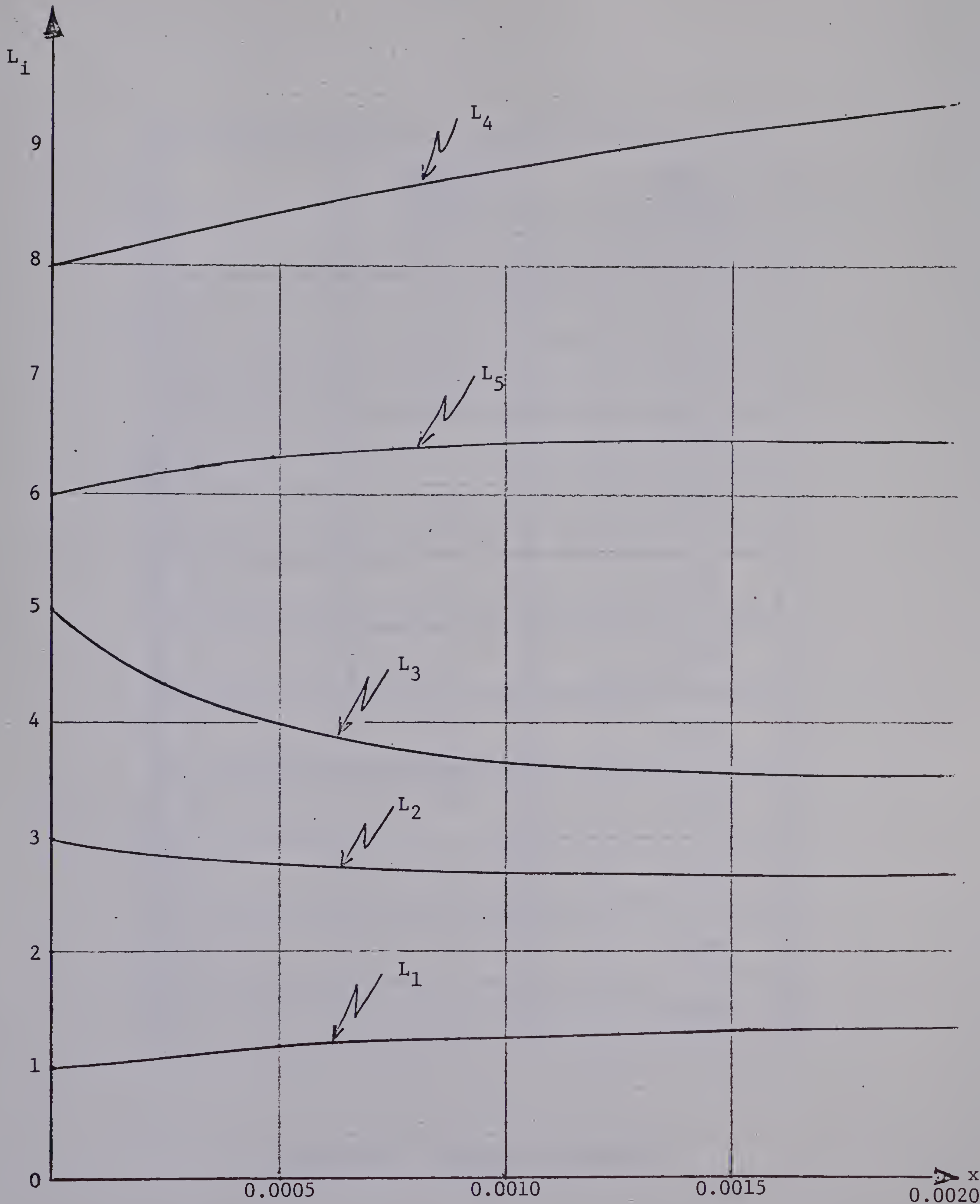


Fig. 2-9. Variation of Inductive Elements with x.

	ORIGINAL NETWORK $x = 0$	NEW NETWORK $x = 0.002$
R_1	3	3.414
R_2	4	3.572
R_3	6	5.013
R_4	3.5	4.544
R_5	7	6.828
L_1	1	1.322
L_2	3	2.670
L_3	5	3.552
L_4	8	9.452
L_5	6	6.486
Φ	11.1	7.573

TABLE 2-3 Summary of Results

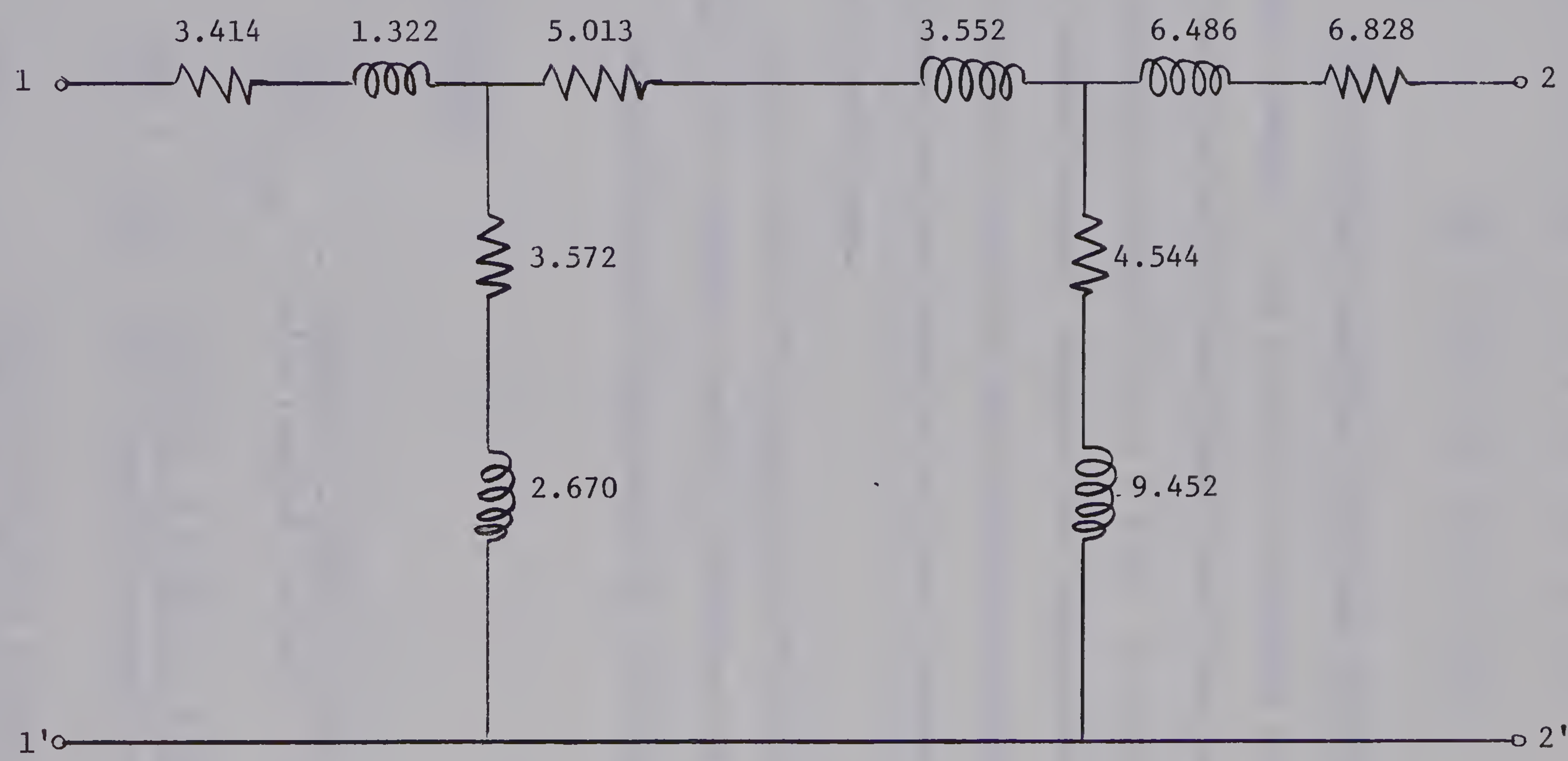


Fig. 2-10 Equivalent Network at $x = 0.002$

$$\underline{R}_b = [R_1 \;\; R_2 \;\; R_3 \;\; R_4 \;\; R_5]^t$$

$$\underline{L}_b = [L_1 \;\; L_2 \;\; L_3 \;\; L_4 \;\; L_5]^t$$

The nonlinear equations (2-34) and (2-35) are solved as before on the digital computer. The solution is displayed in Appendix V. The graphs showing the variations of the elements with x are shown in Figs. 2-8 and 2-9. The solution obtained for $x = 0.002$ is not the ideal optimum, since Φ can be reduced further. However, the computation was stopped at this value of x since it was felt that the distribution of the resistive values was good enough and the values of the other elements were acceptable for purposes of this thesis.

Table 2-3 shows the comparison between the initial values and the new values of the elements, and Fig. 2-10 shows the new network.

The short circuit admittances for the original and new networks are:

Original Network

$$Y(s)_{1-1'}\Big|_{V_2=0} = \frac{0.311\, s^2 + 0.586\, s + 0.252}{s^3 + 3.684\, s^2 + 4.181\, s + 1.436}$$

$$Y(s)_{12} = Y(s)_{21} = \frac{0.047\, s^2 + 0.083\, s + 0.028}{s^3 + 3.684\, s^2 + 4.181\, s + 1.436}$$

$$Y(s)_{2-2'}\Big|_{V_1=0} = \frac{0.107\, s^2 + 0.276\, s + 0.153}{s^3 + 3.684\, s^2 + 4.181\, s + 1.436}$$

Equivalent network

$$Y(s)_{1-1'} = \frac{0.305\, s^2 + 0.572\, s + 0.244}{s^3 + 3.681\, s^2 + 4.175\, s + 1.430}$$

$$Y(s)_{12} = Y(s)_{21} = \frac{0.048\, s^2 + 0.087\, s + 0.031}{s^3 + 3.681\, s^2 + 4.175\, s + 1.430}$$

$$Y(s)_{2-2'} = \frac{0.105\, s^2 + 0.271\, s + 0.150}{s^3 + 3.681\, s^2 + 4.175\, s + 1.430}$$

Table 2-4 shows the percentage deviation of the coefficients of the powers of s in Y(s) of the new network from the corresponding coefficients of the original network.

COEFFICIENT OF	% DEVIATION			
	NUMERATOR OF			DENOMINATOR
	Y_{11}	Y_{12}	Y_{22}	
s^3				0
s^2	1.93	- 2.57	1.87	0.0815
s^1	2.49	- 5.32	1.81	0.144
s^0	3.175	-13.23	1.96	0.418

TABLE 2-4 Percentage Deviation in Coefficients

The large discrepancy in the case of the coefficients of the powers of s in the transfer function seems to result from the fact that in both the original and equivalent networks the coefficients of s and s^0 are very small. Consequently even a small error in their values is magnified when the percentage deviation is computed, since division by a small number is involved.

The question, however, might be asked, Why not compare the open-circuit parameters? If the open-circuit parameters are considered, we shall be dealing with at least one less than the number of nodes we start with, since I_1 or I_2 will have to be assumed equal to zero. Hence in order to take into account all the elements, I_1 and I_2 should be nonzero. One way of keeping I_1 and I_2 nonzero is to short-circuit one of the ports. If this is done, then it is more convenient to talk of short-circuit admittances. This is also according to standard practice.

2-4 Conclusion

It must be pointed out that the equivalent network which is chosen because it is optimal for the spread in the values of the resistances may not be desirable from other points of view. For instance in the two examples considered, we may find that in the "optimal network" the values of inductances may be too large. If the designer is confronted with such a situation, he should perhaps settle for a "sub-optimal" network in which the distribution of resistive values is "good" and the values of inductances are also reasonable. Another approach, a more complicated one, will be to use the comprehensive performance criterion as given by equation (2-1). By a

judicious choice of values of the weighting factors, the relative importance of one or more kinds of elements can be controlled.

Chapter 3

NETWORK OPTIMIZATION--II (Sensitivity)

3-1 Introduction: Recently a number of papers [2, 9, 16] has been published on the application of the concept of continuous equivalence to minimize the sensitivity of network functions to changes in the element values. In this approach the problem of minimization of the sensitivity of a network consists of scanning through a set of networks which are all equivalent, and then picking one with minimum sensitivity. A summary of Schoeffler's approach to this problem is given in Appendix III. It is felt that a simpler approach can be used in most cases except perhaps when the number of elements is unduly large. This procedure will be described in section two of this chapter. Before we do that however, a brief discussion on the classical definition of sensitivity will be presented.

Let T be a network function, e.g., driving point impedance. If we represent the N elements in the network by $e_1, e_2, \ldots\ldots e_K, \ldots\ldots e_N$, then T can be expressed as

$$T = T(s, e_1, e_2, \ldots\ldots, e_N)$$

where s is Laplace transform variable or complex frequency and e_K is any element (L, R or C).

Using the classical definition of sensitivity, the sensitivity S_K, of T to a change in the K^{th} element of the network is given by

$$S_K = \frac{\partial \ln T}{\partial \ln e_K} = \frac{e_K}{T}\frac{\partial T}{\partial e_K}$$

3-2 Minimization of Sensitivity

Let us consider the one-port network in Fig. 3-1 which has $N_R + N_L + N_C$ elements. (N_R, N_L and N_C are the number of resistors,

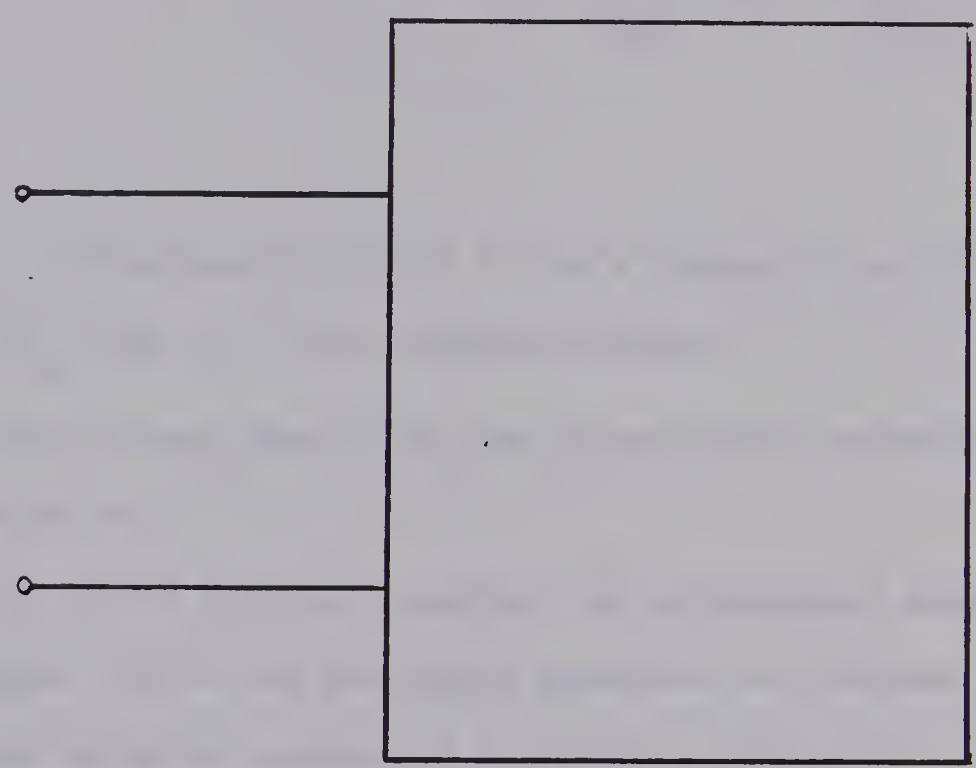
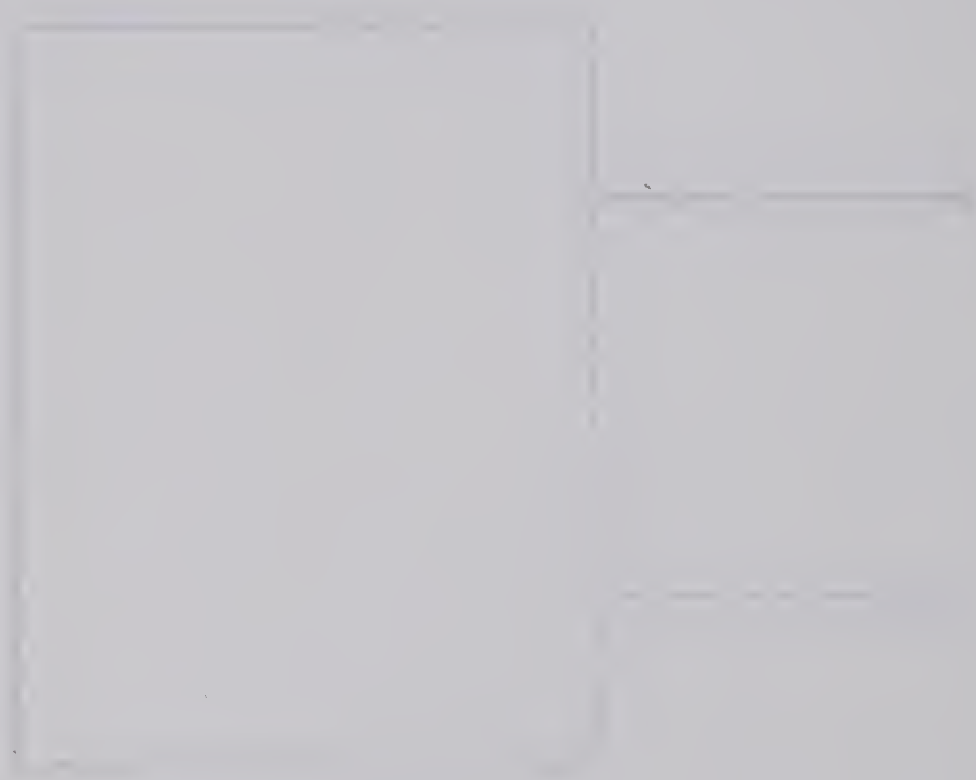


Fig. 3-1 Passive R-L-C Network

inductors and capacitors respectively.) Let us assume that we are required to find its equivalent in which the sensitivity is minimum. Since the sensitivity function S_K is complex, a convenient and meaningful performance criterion as suggested by Schoeffler [16] can be expressed as



$$\Phi = \sum_{K=1}^{N_R+N_L+N_C} \left| S_K \right|^2 \qquad \text{..... (3-2)}$$

$$= \sum_{i}^{N_R} \left| S_{Ri} \right|^2 + \sum_{j=1}^{N_L} \left| S_{Lj} \right|^2 + \sum_{k=1}^{N_C} \left| S_{Ck} \right|^2$$

..... (3-3)

where S_{Ri} is the sensitivity of T to a change in the i^{th} resistive element. S_{Lj} and S_{Ck} are similarly defined.

Our purpose then is to find an equivalent network which has minimum value of Φ.

As in the previous examples, the optimization problem is solved in two stages. First the generating equations are obtained and then Φ is minimized using the generating equations as constraints.

The procedure for deriving the generating equations has been discussed earlier in Chapter 1 (equations 1-22, 1-23, 1-24). We can therefore write the generating equations for the network represented by Fig. 3-1 as

$$\frac{d\underline{R}_b}{dx} = \underline{M}\,\underline{R}_b$$

$$\frac{d\underline{L}_b}{dx} = \underline{M}\,\underline{L}_b \qquad \text{..... (3-4)}$$

$$\frac{d\underline{H}_b}{dx} = \underline{M}\,\underline{H}_b$$

The order of $\underline{M}$ will depend on the actual network configuration, number of branches, etc. The elements of $\underline{M}$ are generally linear combinations of the nonzero elements of $\underline{A}_1$.

For simplicity, let $N_R = N_L = N_C = N$ so that equation (3-2) becomes

$$\Phi = \sum_{K=1}^{3N} \left| S_K \right|^2 \qquad \text{..... (3-5)}$$

As before, the method of steepest descent will be used to ensure that Φ goes rapidly to its minimum. This requires that $\frac{d\Phi}{dx} < 0$

$$\frac{d\Phi}{dx} = \sum_{K=1}^{3N} \frac{\partial \Phi}{\partial e_K} \cdot \frac{de_K}{dx} < 0 \qquad \text{..... (3-6)}$$

where e_K is any element (R, L, or C).

If we substitute for $\frac{de_K}{dx}$, using equation (3-4), in equation (3-6), we shall get an expression of the form

$$\frac{d\Phi}{dx} = \sum_{i=1}^{n} a_i \; f_i \; (e_1, e_2, \ldots.. e_{3N}) < 0 \qquad \text{..... (3-7)}$$

where n is the number of nonzero elements of $\underline{A}_1$ which form the matrix $\underline{M}$.

By defining $a_i = -f_i(e_1, e_2, \ldots.. 3_{3N})$ ..... (3-8)

and substituting in (3-7), we get

$$\frac{d\Phi}{dx} = \sum_{i=1}^{n} - f_i^2 \qquad \text{..... (3-9)}$$

It is easy to see that $\frac{d\Phi}{dx}$ is always negative.

If we then substitute for a_i's as defined by equation (3-8) into equation (3-4), we can solve the resulting nonlinear equation on a digital computer. This method of solution is more straightforward than Schoeffler's approach. It also has the following advantages: (1) Only one set of differential equations will be solved instead of two as Schoeffler's technique requires. (2) Since the solution is self-adjusting, the problem of making an initial and consequent choices of a_i's, every time the generating equations are solved, is eliminated. (3) The problem of putting bounds on a_i's is also eliminated.

In the next section, an example which illustrates the optimization procedure discussed above is presented.

3-3 Example 3-1

Consider the minimization of the sensitivity of the network shown in Fig. 3-2. Since the structure of the network is the same as in Example 2-1, the element generating equations are the same as equations (2-8), and (2-10). The equations are repeated below for convenience.

$$\begin{aligned} \frac{dR_1}{dx} &= - R_2(a_2 + a_1) + R_3a_1 \\ \frac{dR_2}{dx} &= R_2(a_2 - a_1) - R_3a_1 \qquad \text{..... (3-28)} \\ \frac{dR_3}{dx} &= R_2(a_2 + a_1) + R_3(2a_2 + a_1) \end{aligned}$$

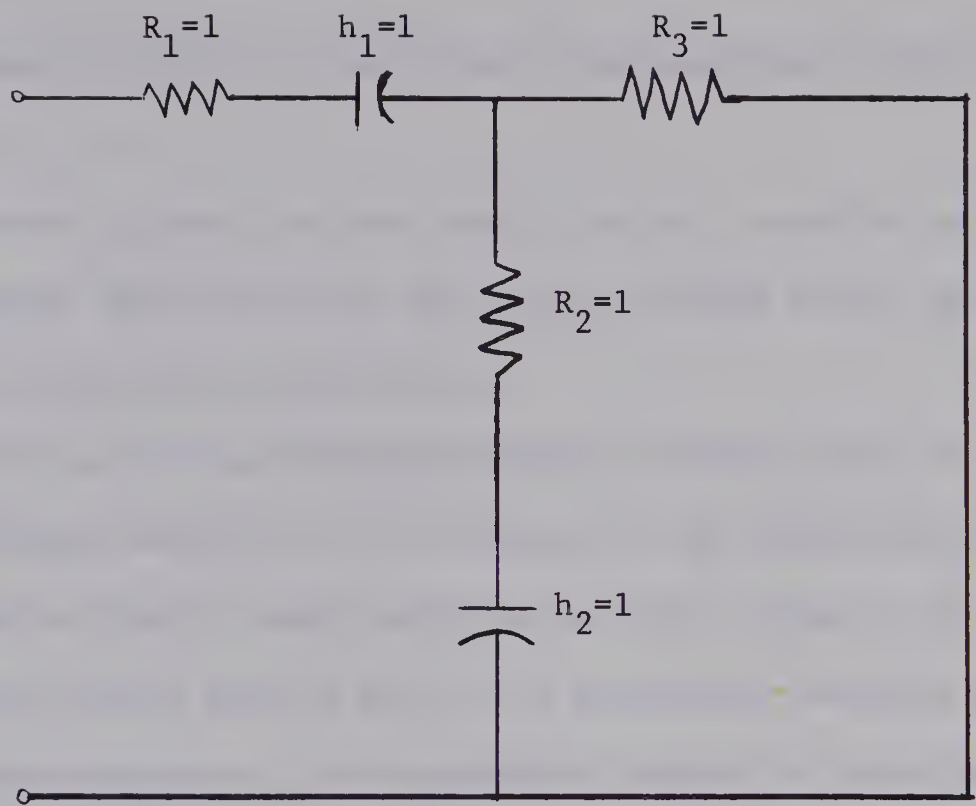


Fig. 3-2 Given R-C Passive Network

$$\frac{dh_1}{dx} = - h_2(a_2 + a_1) + h_3 a_1$$

$$\frac{dh_2}{dx} = h_2(a_2 - a_1) - h_3 a_1 \qquad \text{..... (3-29)}$$

$$\frac{dh_3}{dx} = h_2(a_2 + a_1) + h_3(2a_2 + a_1)$$

where as before

$$h_i = \frac{1}{C_i}$$

We note in Fig. 3-2 that there is no capacitor in the third branch, consequently $h_3 = 0$.

In order to keep the cost down it is our intention to keep the number of elements the same as in Fig. 3-2. In other words, we shall keep h_3 at zero in the equivalent network also.

Before we take up the minimization of sensitivity, we will point out the constraints which have to be imposed on the generating equations in case an element is zero to begin with and we wish to keep it at zero or one of the elements becomes zero in one of the equivalent networks and we do not want it to become negative. The constraints consist of equating the particular element and its derivative with respect to x to zero in the generating equations. Thus in equations (3-29) if $h_3 = 0$ and $\frac{dh_3}{dx} = 0$ we get

$$\frac{dh_1}{dx} = - h_2(a_2 + a_1)$$

$$\frac{dh_2}{dx} = h_2(a_2 - a_1) \qquad \text{..... (3-30)}$$

$$h_2(a_2 + a_1) = 0$$

Since $h_2 \neq 0$

$$a_2 = - a_1 \qquad \text{..... (3-31)}$$

Equation (3-31) gives the additional constraint which should be imposed on equations (3-28) and (3-29) if equivalent network is to result in which $h_3 = 0$.

Substituting equation (3-31) into equations (3-28) and (3-29) we get

$$\frac{dR_1}{dx} = R_3 a_1$$

$$\frac{dR_2}{dx} = -2R_2 a_1 - R_3 a_1$$

$$\frac{dR_3}{dx} = -R_3 a_1 \qquad \text{..... (3-32)}$$

$$\frac{dh_1}{dx} = 0$$

$$\frac{dh_2}{dx} = -2h_2 a_1$$

Now we proceed with the minimization of sensitivity. If we now define the sensitivity criterion Φ as in equation (3-1) our problem is to minimize Φ using equations (3-32) as constraints.

For the example under consideration, Φ is given by

$$\Phi = \sum_{i=1}^{3} \left| S_{R_i} \right|^2 + \sum_{j=1}^{2} \left| S_{h_j} \right|^2 \qquad \text{..... (3-33)}$$

where $S_i = \frac{\partial Z}{\partial e_i} \cdot \frac{e_i}{Z}$

e_i is the i^{th} element R or h

and $\quad Z$ is the driving point impedance.

A calculation of Φ for the network discloses that the series arm or branch one contributes 89.3% of the value of Φ. This confirms Milkurski's [11] claim that elements in series are more effective in their change than those in parallel. Thus, the effect of an element variation on a network function depends a great deal on the position of the element in the network.

In view of this, a simplified version of equation (3-33) can be used such that only the series arm is taken into consideration. The new performance criterion is designated Φ_1 and expressed as:

$$\Phi_1 = \left|S_{R1}\right|^2 + \left|S_{h1}\right|^2 \qquad \ldots\ldots (3\text{-}34)$$

By finding an equivalent network with a reduced value of Φ_1, it is expected that Φ will be reduced. However, due to the nonlinear nature of Φ, it will not change linearly with Φ_1.

The remaining computations will be carried out at a constant value of frequency, namely, $\omega = 1$ radian per second. (In other words, we have let $s = j\omega = j1$ in equation (3-34).)

Equation (3-34) can be written as

$$\Phi_1 = \frac{(R_1^2 + h_1^2)\,\{(R_2 + R_3)^2 + h_2^2\}}{\{[h_1h_2 - (R_1R_2 + R_1R_3 + R_2R_3)]^2 + (h_1R_2 + h_1R_3 + h_2R_1 + h_2R_3)^2\}} \qquad \ldots\ldots (3\text{-}35)$$

For simplification purposes, the following definitions are made:

$$K_{11} = (R_2 + R_3)^2 + h_2^2$$

$$K_{12} = R_1^2 + h_1^2$$

$$K_{13} = h_1 h_2 - (R_1 R_2 + R_1 R_3 + R_2 R_3)$$

$$K_{14} = h_1 R_2 + h_1 R_3 + h_2 R_1 + h_2 R_3$$

$$K_{15} = K_{13}^2 + K_{14}^2$$

$$K_{16} = K_{11}\, K_{12}$$

$$K_{17} = 2K_{11}\, K_{15}\, R_1$$

$$K_{18} = 2K_{16}\, \{K_{14} h_2 - K_{13}(R_2 + R_3)\}$$

$$K_{19} = 2K_{11}\, K_{15}\, h_1$$

$$K_{20} = 2K_{16}\, \{K_{13} h_2 + K_{14}(R_2 + R_3)\}$$

$$K_{21} = 2K_{12}\, K_{15}(R_2 + R_3)$$

$$K_{22} = 2K_{16}\, \{K_{14} h_1 - K_{13}(R_1 + R_3)\}$$

$$K_{23} = 2K_{16}\, \{K_{14}(h_1 + h_2) - K_{13}(R_1 + R_2)\}$$

$$K_{24} = 2K_{12}\, K_{15}\, h_2$$

$$K_{25} = 2K_{16}\, \{K_{13} h_1 + K_{14}(R_1 + R_3)\}$$

$$K_{26} = R_3(K_{17} + K_{22} + K_{23} - K_{18} - 2K_{11})$$

$$K_{27} = 2R_2(K_{22} - K_{21})$$

$$K_{29} = 2h_2(K_{25} - K_{24})$$

$$K_{30} = K_{26} + K_{27} + K_{29}$$

..... (3-36)

Applying the method of steepest descent on Φ_1 to arrive at its minimum requires

$$\frac{d\Phi_1}{dx} < 0 \qquad \text{..... (3-37)}$$

Substituting equation (3-35) in (3-37) and using the definitions in (3-36) the result is:

$$a_1 \cdot \frac{K_{30}}{K_{15}^2} < 0 \qquad \text{..... (3-38)}$$

One way of satisfying inequality (3-38) is to define a_1 as:

$$a_1 = \frac{-K_{30}}{K_{15}^2} \qquad \text{..... (3-39)}$$

Although K_{15}^2 is obviously positive, it is retained in equation (3-39) because (as it was discovered during numerical computations) K_{30} may be too large for the computer to handle. Retaining K_{15}^2

enables us to control the value of a_1 since K_{15} like K_{30} depends on the values of the elements of the network.

Substituting for a_1 in equation (3-32) the final generating equations are:

$$\frac{dR_1}{dx} = - R_3 \cdot \frac{K_{30}}{K_{15}^2}$$

$$\frac{dR_2}{dx} = R_2 \cdot \frac{2K_{30}}{K_{15}^2} + R_3 \cdot \frac{K_{30}}{K_{15}^2}$$

$$\frac{dR_3}{dx} = R_3 \cdot \frac{K_{30}}{K_{15}^2} \qquad \text{..... (3-40)}$$

$$\frac{dh_1}{dx} = 0$$

$$\frac{dh_2}{dx} = h_2 \cdot \frac{2K_{30}}{K_{15}^2}$$

A part of the solution of equation (3-40) is displayed in Appendix VI and the curves representing the variations of the elements with x are shown in Fig. 3-3.

Assuming that we are satisfied with the value of Φ at $x = 0.177$, the network that results is shown in Fig. 3-4.

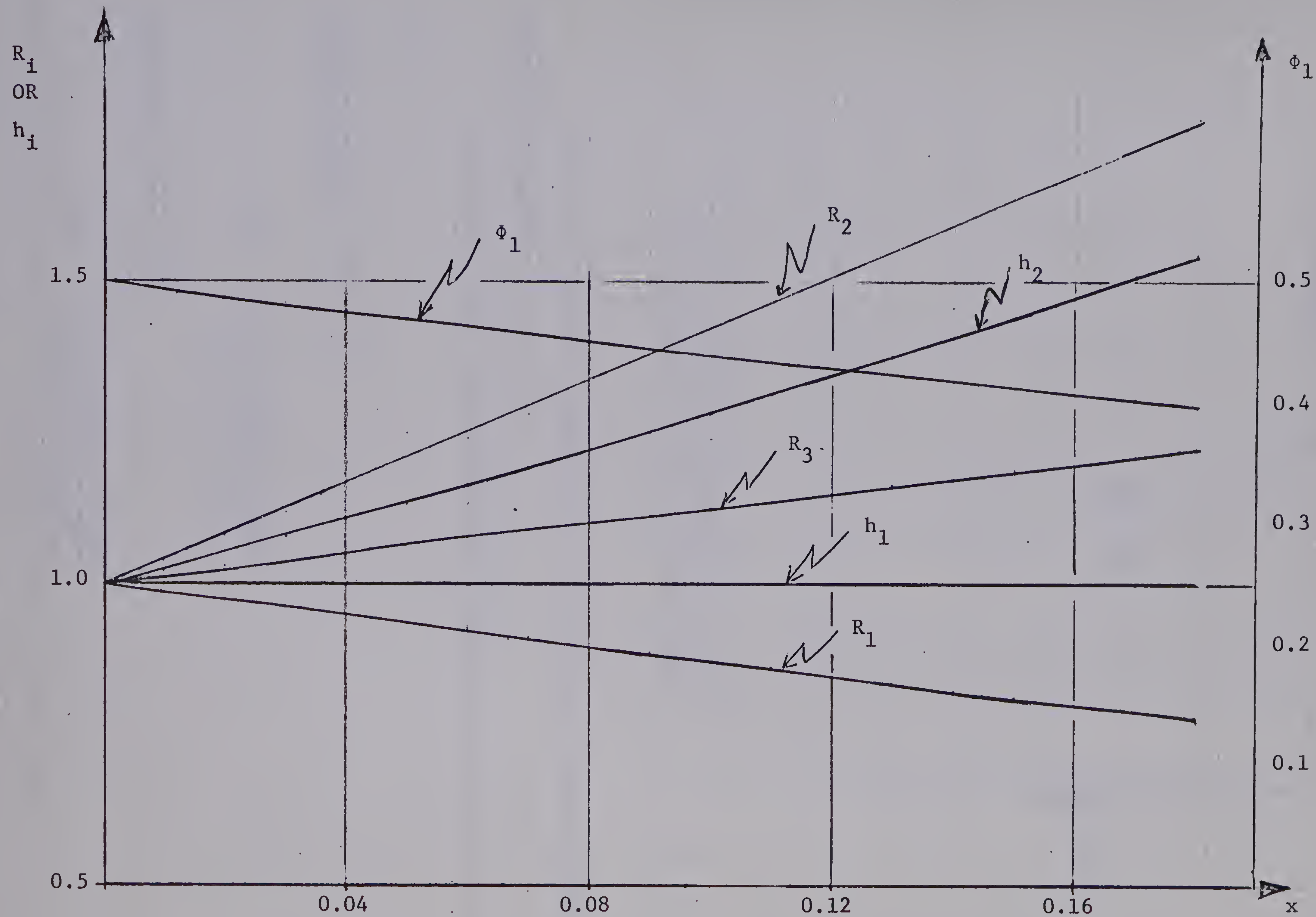


Fig. 3-3 Variation of the Network Elements with x.

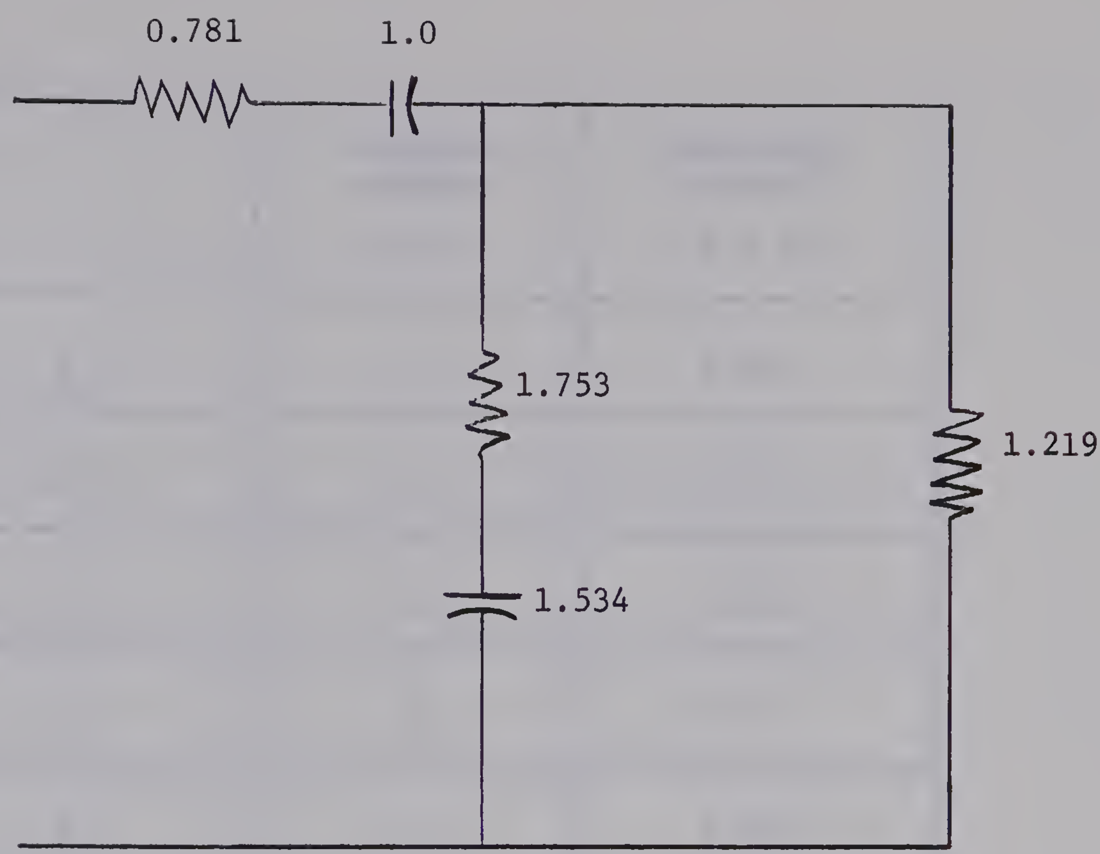


Fig. 3-4 Equivalent Network at x = 0.177
R is in ohms and C is in inverse farad.

Table 3-1 shows the comparison of the different network elements and their respective Φ at $\omega = 1$.

The driving point impedances of the original and the new networks are:

<u>Original Network</u> (Fig. 3-2)

$$Z(s) = \frac{s^2 + 1.333\ s + 0.333}{0.667\ s^2 + 0.333\ s}$$

<u>New Network</u> (Fib. (3-3)

$$Z(s) = \frac{s^2 + 1.325\ s + 0.337}{0.625\ s^2 + 0.337\ s}$$

	ORIGINAL NETWORK $x = 0$	EQUIVALENT NETWORK $x = 0.177$
R_1	1	0.781
R_2	1	1.753
R_3	1	1.219
h_1	1	1.0
h_2	1	1.534
h_3	0	0
Φ	0.56	0.51

TABLE 3-1 Summary of Results

Table 3-2 shows the percentage deviation of the coefficients of the powers of s in the $Z(s)$ of the new network from the corresponding coefficients of the original network.

COEFFICIENTS OF	% DEVIATION	
	NUMERATOR	DENOMINATOR
s^2	0	2.1
s^1	0.6	- 1.333
s^0	- 1.333	---

TABLE 3-2 Percentage Deviation in Coefficients

It should be mentioned at this point that the new network represented at $x = 0.177$ is by no means the minimum sensitivity network since the value of Φ can be reduced further. However, the solution was stopped at $x = 0.177$ since the main objective of this discussion is to establish the usefulness of the procedure. As stated earlier, the choice of a network depends on several factors and the designer will have to make the decision. Some of these factors are discussed in Chapter 4.

3-4 Comments

Although the minimization of sensitivity was carried out with respect to the elements in the series branch, this procedure can equally well be used to minimize the sensitivity with respect to any other elements or groups of elements in the network.

Schoeffler [16] and Leed [9] have considered examples which lead to the conclusion that the equivalent network with minimum sensitivity has more elements than the original network. If their procedure is applied to the example discussed in this chapter, the minimum sensitivity network may turn out to be one with at least six elements. The minimum value of sensitivity of the six-element network may be less than the minimum value for the five-element network. Even if this is so, it must be remembered that the advantage gained by reducing sensitivity may be offset by the increase in cost due to the additional elements. A judgement will have to be made in each individual case.

Chapter 4

SUMMARY AND CONCLUSION

4-1 Some Limitations of Continuous Equivalence Theory

Although Schoeffler [15] published his theory of continuously equivalent networks in 1964, only during the past couple of years have reports on some of the limitations of the theory appeared in the literature. In 1966, Newcomb [12] produced an example of two networks with the same configuration, the same number of elements and the same driving point impedance. He showed that neither of these networks could be obtained from the other by Schoeffler's method. Thus, he concluded that not all the equivalent networks can be found by Schoeffler's method. Newcomb arrived at this conclusion by solving the generating equations.

By way of amplifying Newcomb's statement, it must be pointed out that other equivalent networks of either of the two networks considered by Newcomb can be generated by Schoeffler's method provided we are allowed to increase the number of elements. Another point which may be worth mentioning is that it is not always necessary to solve the generating equations, as Newcomb did, to conclude whether Schoeffler's theory fails or not in a particular case.

In order to illustrate this point, consider the following example:

Example 4-1

Suppose we are given the network in Fig. 4-1 (this is the same network as in example 2-1 except that $R_2 = 0$ and $h_3 = 0$) and we are asked to find an equivalent network with the same configuration and same number

of elements.

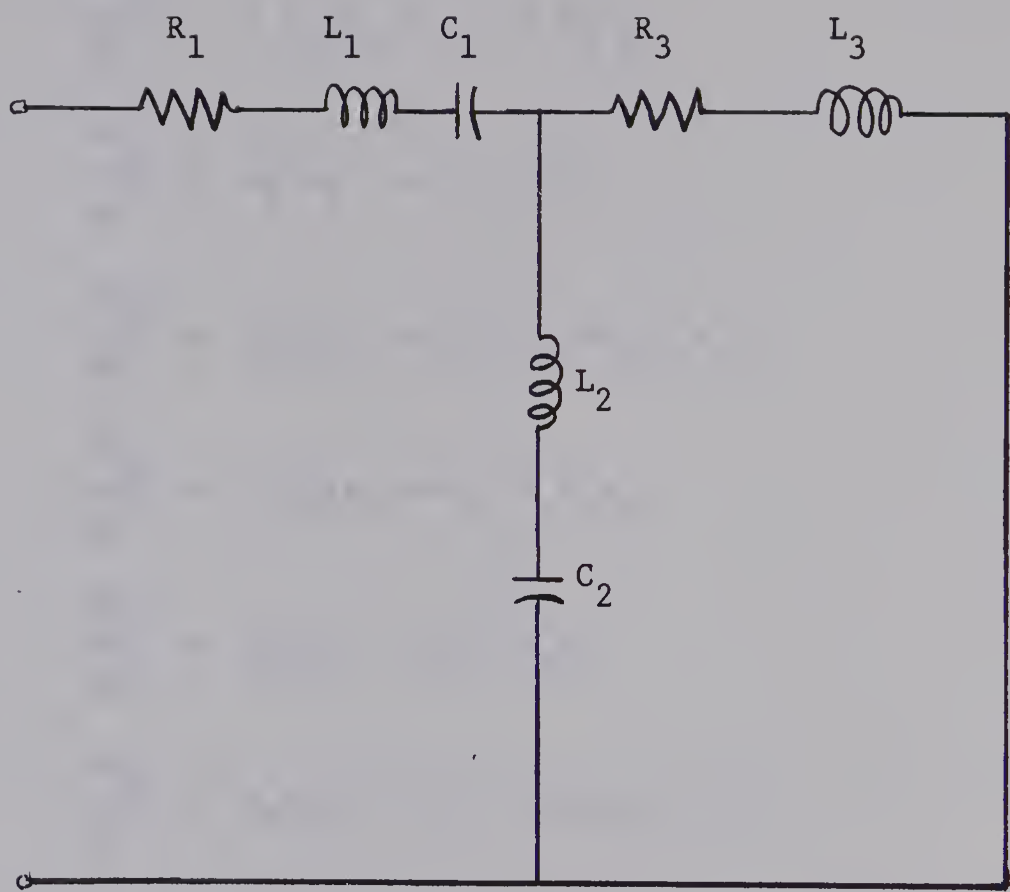


Fig. 4-1 One-Port R-L-C.

Dimensions are in ohms, henry and daraf.

We can use the generating equations obtained for example 2-1 but must allow for the fact that $R_2 = 0$ and $h_3 = 0$. The equations are:

$$\frac{dR_1}{dx} = -R_2(a_2 + a_1) + R_3 a_1$$

$$\frac{dR_2}{dx} = R_2(a_2 - a_1) - R_3 a_1$$

$$\frac{dR_3}{dx} = R_2(a_2 + a_1) + R_3(2a_2 + a_1)$$

$$\frac{dL_1}{dx} = - L_2(a_2 + a_1) + L_3a_1$$

$$\frac{dL_2}{dx} = L_2(a_2 - a_1) - L_3a_1$$

$$\frac{dL_3}{dx} = L_2(a_2 + a_1) + L_3(2a_2 + a_1)$$

$$\frac{dh_1}{dx} = - h_2(a_2 + a_1) + h_3a_1$$

$$\frac{dh_2}{dx} = h_2(a_2 - a_1) - h_3a_1$$

$$\frac{dh_3}{dx} = h_2(a_2 + a_1) + h_3(2a_2 + a_1)$$

..... (4-1)

Since $R_2 = 0$ and $h_3 = 0$, we impose the constraints

$$\frac{dR_2}{dx} = 0; \quad R_2 = 0$$

..... (4-2)

$$\frac{dh_3}{dx} = 0; \quad h_3 = 0$$

on equation (4-1).

The generating equations become

$$\frac{dR_1}{dx} = 0$$

$$\frac{dR_3}{dx} = 0$$

$$\frac{dL_1}{dx} = 0$$

$$\frac{dL_2}{dx} = 0$$

$$\frac{dL_3}{dx} = 0$$

$$\frac{dh_1}{dx} = 0$$

$$\frac{dh_2}{dx} = 0$$

..... (4-3)

Since equations (4-3) imply that all the elements are constant, it can be concluded that Schoeffler's method will not yield a network equivalent to Fig. 4-1, if the same configuration and number of elements is to be maintained. It may, however, be possible to find an equivalent network if the number of elements can be increased since this will mean the deletion of the constraints given by equation (4-2).

Another situation where Schoeffler's method fails is related to multiport networks. In view of the fact that Schoeffler's method is an extension of Howitt's work, one of the conditions which must be satisfied for Schoeffler's method to be valid is that the number of loops must be greater than the number of ports. There is no provision in Schoeffler's method to alter the number of loops in the course of the solution. For instance, Schoeffler's method cannot be used to find an equivalent of the two-port

network shown in Fig. 4-2 because the network has only two loops.

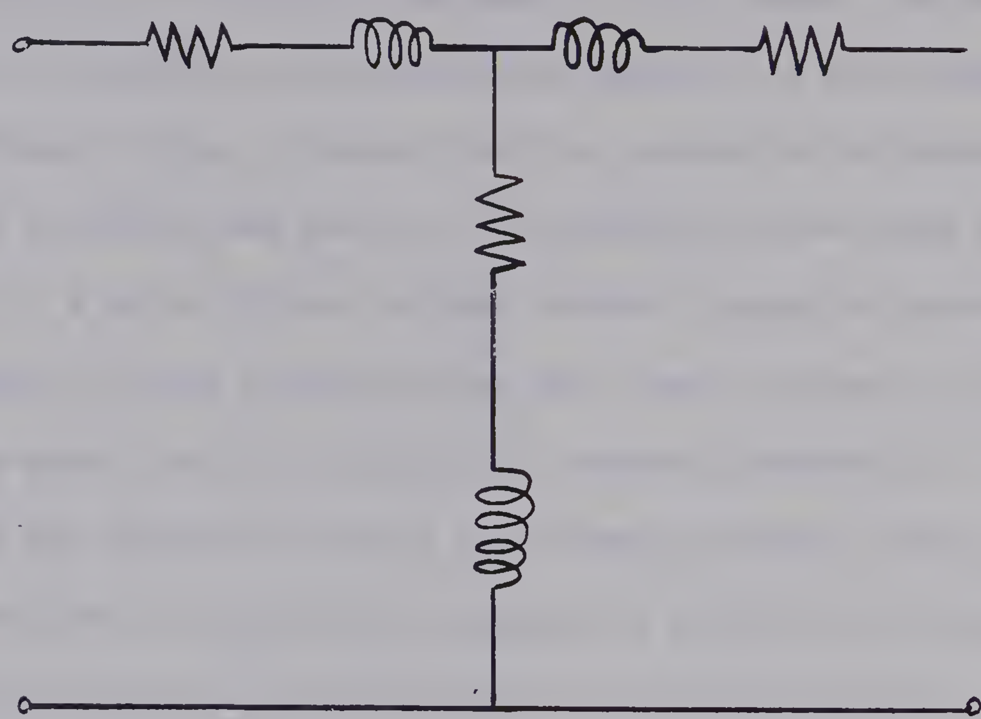


Fig. 4-2 Two-Port R-L Network.

Lastly, the network shown in Fig. 4-3 is another example in which Schoeffler's theory does not yield an equivalent. The reason for the non-equivalence for this case is still under investigation. Analytically, the theory is supposed to yield an equivalent, but computer solution has revealed that the deviation of the driving point impedance of the equivalent network from that of the original network increases as x increases.

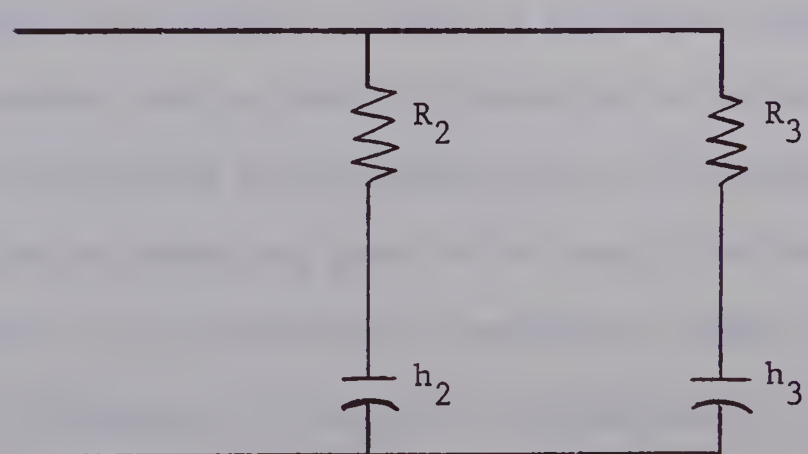


Fig. 4-3 One-Port R-C Network.

4-2 Comments on the Choice of Optimal Networks

In all the examples discussed in this thesis, caution has been exercised in referring to any particular solution as the optimal solution. The main reason is that a network which is optimal with respect to one performance criterion may prove to be undesirable from other points of view. The choice of a truly optimal network depends largely on practical considerations. Some of these considerations are: small values of capacitances are undesirable since they are difficult to measure accurately; large values of inductances are affected by shunt capacitances between their windings. It is therefore left to the network designer to exercise his discretion based on practical experience to choose what he considers to be the optimal network.

4-3 Areas for Further Study

As pointed out earlier, Schoeffler's method may not yield all the equivalents of a given network or may yield none at all. At the present time, in order to determine whether Schoeffler's method will work or not, one has to obtain the generating equations which in itself may be a time-consuming task especially if the network has several elements. Therefore it would appear that one of the areas for further research should deal with the development of simple procedures or tests to determine whether the equivalent of a given network can be found by Schoeffler's method.

If the limitations of the method can be eliminated, Schoeffler's continuous equivalence theory may prove to be one of the best tools available to network designers for optimization of networks. Hence, further research is worth while to eliminate, if possible, at least some of the limitations.

APPENDIX I

Theorem: Two $(n + m) \times (n + m)$ loop-impedance matrices $Z_a(s)$ and $Z_b(s)$ have the same $n \times n$ terminal-impedance matrix $Z(s)$ if

$$\underline{Z}_b = \underline{\mathcal{A}}\, \underline{Z}_a \underline{A} \qquad \text{..... (A1-1)}$$

where

$$\underline{\mathcal{A}} = \begin{array}{cc} \left[\begin{array}{c|c} \underline{u}_n & \underline{\alpha}_{12} \\ \hline \underline{0} & \underline{\alpha}_{21} \end{array}\right] & \begin{array}{l} \}\ n \\ \}\ m \end{array} \\ \begin{array}{cc} \underbrace{\quad}_{n} & \underbrace{\quad}_{m} \end{array} & \end{array} \qquad \text{..... (A1-2)}$$

$$\underline{A} = \begin{array}{cc} \left[\begin{array}{c|c} \underline{u}_n & \underline{0} \\ \hline \underline{a}_{21} & \underline{a}_{22} \end{array}\right] & \begin{array}{l} \}\ n \\ \}\ m \end{array} \\ \begin{array}{cc} \underbrace{\quad}_{n} & \underbrace{\quad}_{m} \end{array} & \end{array} \qquad \text{..... (A1 - 3)}$$

and $\underline{\mathcal{A}}$ is assumed nonsingular.

<u>Proof:</u>

Consider the $(n + m)$-port described by Z_b. Short-circuiting the final m ports yields the description

$$\begin{array}{cc} \left[\begin{array}{c} \underline{V}_1 \\ \hline \underline{0} \end{array}\right] & \begin{array}{l} \}\ n \\ \}\ m \end{array} \end{array} = \underline{Z}_b \left[\begin{array}{c} \underline{I}_1 \\ \hline \underline{I}_2 \end{array}\right] = \underline{\mathcal{A}}\, \underline{Z}_a \underline{A} \left[\begin{array}{c} \underline{I}_1 \\ \hline \underline{I}_2 \end{array}\right] \qquad \text{..... (A1-4)}$$

where the voltage and current vectors are partitioned as the ports.

Premultiplying equation (A1-4) by $\underline{\mathcal{A}}^{-1}$ we get

$$\begin{bmatrix} \underline{V}_1 \\ \underline{0} \end{bmatrix} = \underline{Z}_a \begin{bmatrix} \underline{I}_1 \\ \underline{I}_2' \end{bmatrix}$$

where $\underline{I}_2' = \underline{a}_{21}\underline{I}_1 + \underline{a}_{22}\underline{I}_2$

Since the variables V_1 and I_1 for all I_1 are the same for the two circuits, they must be described by the same terminal impedance matrix,

$$\underline{V}_1 = \underline{Z}\underline{I}_1$$

Q.E.D.

The theorem and its proof is taken from Newcomb [13], but the notations have been changed to conform with the notations in this thesis.

APPENDIX II

A summary of Schoeffler's [15] development of continuously equivalent networks is presented in this appendix. Some of the notations have been changed for clarity.

In order to formulate the problems of continuously equivalent networks, consider the transformation matrix $\underline{T}$ which transforms the branch admittance matrix $\underline{Y}_b$ into $\underline{Y}_b + \Delta\underline{Y}_b$ where $\Delta\underline{Y}_b$ is incremental in size.

$$\text{Let} \quad \underline{T} = \underline{u} + \Delta\underline{T} \qquad \text{..... (A2-1)}$$

where $\underline{u}$ is $b \times b$ unit matrix and b is the number of branches.

The application of $\underline{T}$ to $\underline{Y}_b$ results in $\underline{Y}_b'$ given by

$$\underline{Y}_b' = \underline{T}' \, \underline{Y}_b \, \underline{T} \qquad \text{..... (A2-2)}$$

$$= \underline{Y}_b + \underline{Y}_b \Delta T + \Delta\underline{T}^t \, \underline{Y}_b + \Delta\underline{T}^t \, \underline{Y}_b \Delta T \qquad \text{..... (A2-3)}$$

define

$$\Delta\underline{T} = \underline{T}_1 \Delta x \qquad \text{..... (A2-4)}$$

where Δx is a scalar.

Then $\underline{T}$ and $\underline{Y}_b$ become

$$\underline{T} = \underline{u} + \Delta\underline{T} = \underline{u} + \underline{T}_1 \Delta x \qquad \text{..... (A2-5)}$$

$$\underline{Y}_b' = \underline{Y}_b + (\underline{Y}_b \, \underline{T}_1 + T_1^t \, \underline{Y}_b) \Delta x + o(\Delta x^2) \qquad \text{..... (A2-6)}$$

Writing $\underline{T}_1 = \underline{T}_1(x)$, $\underline{Y}_b = \underline{Y}_b(x)$ and $\underline{Y}_b' = \underline{Y}_b(x + \Delta x)$ equation (A2-6) becomes

$$\frac{\underline{Y}_b(x + \Delta x) - \underline{Y}_b(x)}{\Delta x} = \underline{Y}_b \, \underline{T}_1 + \underline{T}_1^t \, \underline{Y}_b + \frac{0(\Delta x^2)}{\Delta x} \qquad \text{..... (A2-7)}$$

Passing to the limit yields the matrix differential equation

$$\frac{d\underline{Y}_b}{dx} = \underline{Y}_b \, \underline{T}_1 + \underline{T}_1^t \, \underline{Y}_b \qquad \text{..... (A2-8)}$$

The matrix $\underline{T}_1(x)$ must be selected so that equivalent networks result. To derive the equivalence restrictions on T_1, let us start with the general restriction

$$\underline{T}\underline{\alpha}^t = \underline{\alpha}^t \underline{A} \qquad \text{..... (A2-9)}$$

where α is the usual $n \times b$ cut set matrix of the network, $\underline{A}$ is Cauer transformation matrix, and n is the number of independent node pairs.

Writing

$$\underline{T} = \underline{u} + \underline{T}_1 \Delta x \qquad \text{..... (A2-10)}$$

and

$$\underline{A} = \underline{u} + \underline{B}\Delta x \qquad \text{..... (A2-11)}$$

equation (A2-9) becomes

$$\underline{\alpha}^t + \underline{T}_1\underline{\alpha}^t\Delta x = \underline{\alpha}^t + \underline{\alpha}^t\underline{B}\Delta x \qquad \text{..... (A2-12)}$$

$$\text{or} \quad \underline{T}\underline{\alpha}^t = \underline{\alpha}^t\underline{B} \qquad \text{..... (A2-13)}$$

where $\underline{B}$ is of the form (considering the preservation of the driving point and transfer impedances at terminal pairs 1 and 2)

$$\underline{B} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ x & x & x & x \\ x & x & x & x \end{bmatrix} \qquad \text{..... (A2-14)}$$

For the most general branch in which R, L and C are connected in parallel,

$$\underline{Y}_b = \underline{G} + s\underline{C} + \frac{1}{s}\underline{L}^{-1} \qquad \text{..... (A2-15)}$$

where $\underline{G}$ is the branch conductance matrix

$\underline{C}$ is the branch capacitance matrix

and $\underline{L}^{-1}$ is the branch inverse inductance matrix.

Thus, the transformation $\underline{T}$ can be applied to each element kind separately provided the same Cauer transformation matrix $\underline{A}$ results for each element kind. That means the branch admittance matrices may be transformed as

$$\underline{G}' = \underline{T}_{G1}^t\ \underline{G}\ \underline{T}_{G1} \qquad \text{..... (A2-16)}$$

$$\underline{C}' = \underline{T}_{C1}^t\ \underline{C}\ \underline{T}_{C1} \qquad \text{..... (A2-17)}$$

$$(\underline{L}^{-1})' = \underline{T}_{L1}^t\ \underline{L}^{-1}\ \underline{T}_{L1} \qquad \text{..... (A2-18)}$$

provided

$$\underline{T}_{G1}\underline{\alpha}^t = \underline{T}_{C1}\underline{\alpha}^t = \underline{T}_{L1}\underline{\alpha}^t = \underline{\alpha}^t\underline{B}$$

APPENDIX III

Schoeffler [16] presents an approach to the synthesis of minimum sensitivity networks. Below is a summary of the relevant equations which lead to the minimum sensitivity network.

The generating equations are generally of the form

$$\frac{d\underline{G}}{dx} = \underline{M}\underline{G}$$

$$\frac{d\underline{C}}{dx} = \underline{M}\underline{C} \qquad \text{..... (A3-1)}$$

$$\frac{d\underline{L}}{dx} = \underline{M}\underline{L}$$

where $\underline{G}$ is the branch conductance matrix

$\underline{C}$ is the branch capacitance matrix

and $\underline{L}$ is the branch inductance matrix.

Define the sensitivity of, say, the transfer function T to a change in the k^{th} element e_k by

$$S_k = \frac{e_k}{T} \cdot \frac{\partial T}{\partial e_k} \qquad \text{..... (A3-2)}$$

Also define

$$S_k = q_k \, e_k \qquad \text{..... (A3-3)}$$

After some mathematical manipulations, Schoeffler arrives at the equations for all the sensitivities in matrix form given by:

$$\frac{dQ_G}{dx} = -M^t Q_G$$

$$\frac{dQ_C}{dx} = -M^t Q_C \qquad \text{..... (A3-4)}$$

$$\frac{dQ_L}{dx} = -M^t Q_L$$

where $$Q_G^t = [q_{G1}, q_{G2} \; ..... \; q_{Gb}]$$

Q_C and Q_L can be similarly defined, and t indicates the transpose.

Define the performance criterion Φ by:

$$\Phi = \frac{1}{2} \sum_k [S_{Gk} S^*_{Gk} + S_{Ck} S^*_{Ck} + S_{Lk} S^*_{Lk}] \qquad \text{..... (A3-5)}$$

where * indicates conjugate.

$$\begin{aligned}\frac{d\Phi}{dx} = & - Q_G^t [D_G^2 M^t + MD_G^2] Q_C^* + G^t D_{QG} D^*_{QG} MG \\ & - Q_C^t [D_C^2 M^t + MD_C^2] Q_C^* + C^t D_{QC} D^*_{QC} MC \qquad \text{..... (A3-6)} \\ & - Q_L^t [D_L^2 M^t + MD_L^2] Q_L^* + L^t D_{QL} D^*_{QL} ML\end{aligned}$$

where the Q_G, Q_C, Q_L, G, C and L are column matrices whose elements are the various q_k and the values of the elements of the network and where the D matrices are diagonal matrices defined by:

$$D_G = \text{Diag } [g_1, g_2 \; ..... \; g_b]$$

$$D_{QG} = \text{Diag } [q_{G1}, q_{G2} \; ..... \; q_{Gb}]$$

For steepest descent to the minimum, $\frac{d\Phi}{dx}$ should be as negative as possible. Thus, the free elements b_i ($|b_i| \leq 1$) which appear in M should be chosen to make this so.

"The optimization of the network then proceeds by determining $\frac{d\Phi}{dx}$, choosing the b's, calculating the transformation matrix M, transforming to another equivalent network by integrating equations (A3-1) and (A3-4), choosing another set of b's, etc., until the minimum is obtained.

APPENDIX IV

x	R_1	R_2	R_3	L_1	L_2	L_3	$\frac{1}{C_1}$	$\frac{1}{C_2}$	$\frac{1}{C_3}$	Φ
0	3	2	4	1	2	3	2	1	0.5	2
0.0025	2.982	2.214	3.466	1.029	2.176	2.536	2.059	1.041	0.372	0.797
0.0050	2.980	2.351	3.191	1.053	2.289	2.299	2.095	1.070	0.307	0.382
0.0075	2.978	2.449	3.037	1.068	2.370	2.167	2.117	1.092	0.269	0.210
0.0100	2.973	2.523	2.950	1.076	2.433	2.091	2.131	1.110	0.247	0.128
0.0125	2.967	2.582	2.900	1.078	2.483	2.047	2.140	1.125	0.233	0.085
0.0150	2.960	2.630	2.873	1.077	2.524	2.022	2.145	1.138	0.225	0.059
0.0175	2.953	2.670	2.860	1.075	2.558	2.009	2.148	1.150	0.220	0.042
0.0200	2.945	2.703	2.853	1.071	2.586	2.003	2.150	1.159	0.217	0.030
0.0225	2.938	2.731	2.852	1.067	2.611	2.001	2.151	1.167	0.216	0.022
0.0250	2.932	2.754	2.854	1.063	2.631	2.001	2.151	1.174	0.215	0.016
0.0275	2.926	2.774	2.856	1.059	2.649	2.003	2.151	1.180	0.215	0.012
0.0300	2.921	2.791	2.860	1.056	2.664	2.005	2.151	1.186	0.215	0.008

Computer Result for Example 2-1.

(The actual computation was carried out at intervals of $\Delta x = 0.0005$. For the sake of brevity only certain values of x are displayed.)

APPENDIX V

ELEMENTS

x	R_1	R_2	R_3	R_4	R_5	L_1	L_2	L_3	L_4	L_5	Φ
0	3	4	6	3.5	7	1	3	5	8	6	11.8
0.00025	3.150	3.868	5.503	3.713	6.989	1.117	2.898	4.358	8.250	6.221	9.937
0.00050	3.242	3.775	5.250	3.886	6.970	1.189	2.827	4.013	8.477	6.344	9.071
0.00075	3.302	3.708	5.118	4.034	6.947	1.236	2.775	3.815	8.684	6.412	8.574
0.00100	3.342	3.659	5.050	4.163	6.921	1.267	2.737	3.699	8.873	6.449	8.244
0.00125	3.370	3.623	5.019	4.276	6.895	1.288	2.710	3.631	9.043	6.468	8.006
0.00150	3.390	3.598	5.008	4.377	6.870	1.303	2.691	3.590	9.196	6.478	7.826
0.00175	3.404	3.582	5.007	4.466	6.847	1.314	2.678	3.566	9.332	6.483	7.685
0.00200	3.414	3.572	5.013	4.544	6.828	1.322	2.670	3.552	9.452	6.486	7.573

Computer Result for Example 2-2.

(The actual computation was carried out at intervals of $\Delta x = 0.00005$.)

APPENDIX VI

NETWORK ELEMENTS

x	R_1	R_2	R_3	h_1	h_2	Φ_1
0	1	1	1	1.000	1	0.500
0.010	0.987	1.039	1.013	1.000	1.026	0.494
0.020	0.975	1.078	1.025	1.000	1.052	0.487
0.030	0.962	1.117	1.038	1.000	1.079	0.481
0.040	0.949	1.157	1.051	1.000	1.107	0.475
0.050	0.937	1.198	1.063	1.000	1.135	0.469
0.060	0.924	1.239	1.076	1.000	1.163	0.463
0.070	0.912	1.281	1.088	1.000	1.192	0.457
0.080	0.899	1.323	1.101	1.000	1.222	0.451
0.090	0.887	1.365	1.113	1.000	1.252	0.446
0.100	0.874	1.408	1.126	1.000	1.283	0.440
0.110	0.862	1.452	1.138	1.000	1.314	0.435
0.120	0.850	1.496	1.150	1.000	1.345	0.429
0.130	0.838	1.540	1.162	1.000	1.377	0.424
0.140	0.825	1.584	1.174	1.000	1.410	0.419
0.150	0.813	1.629	1.187	1.000	1.443	0.413
0.160	0.801	1.675	1.199	1.000	1.476	0.408
0.170	0.789	1.721	1.211	1.000	1.510	0.403
0.177	0.781	1.753	1.219	1.000	1.533	0.400
0.180	0.777	1.767	1.223	1.000	1.544	0.398

Computer Solution for Example 3-1.

(Actual computation carried out at intervals of $\Delta x = 0.001$.)

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